Homework 2: Predicate Logic

Due date: Friday October 16 at 11:59 PM (Seattle time, i.e. GMT-7)
If you work with others (and you should!), remember to follow the collaboration policy.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting.

Be sure to read the grading guidelines for more information on what we're looking for.

1. Circuit du Soleil [10 points]

In this problem, we will design a circuit to compute the function M defined by:

$$M(p, q, 1) := p$$

$$M(p, q, 0) := \neg q$$

We'll design the circuit by first coming up with two propositional formulas involving subsets of p, q, r that correspond to the cases above.

(a) Give a propositional logic formula which evaluates to p when r is true and evaluates to false when r is false. [2 points]

(b) Give a propositional logic formula which evaluates to $\neg q$ when r is false and evaluates to false when r is true. [2 points]

(c) Draw a circuit that takes p, q, r as input, uses only AND,OR, and NOT gates, and outputs $M(p, q, r)$. Your gates should not take more than two inputs. (combine your answers from (a) and (b)!) [6 points]

2. Think Contrapositive Be Contrapositive [21 points]

(a) If I go to the store, then I will cook for myself and eat nachos.

   (i) convert this sentence to propositional logic (as on homework 1, ensure you're giving variables to atomic propositions, not compound ones). [2 points]

   (ii) take the contrapositive symbolically, and simplify so that $\neg$ signs are next to atomic propositions (i.e. only single variables). [2 points]

   (iii) translate the contrapositive back to English. [3 points]

(b) In order to get on the lightrail, it is necessary to buy a ticket.

   Repeat steps (i)-(iii) from (a) for this sentence.

(c) If Robbie is on the bus, then if he has yarn he will knit something.

   Repeat steps (i)-(iii) from (a) for this sentence. Note that this statement has two implications; take the contrapositive of the "outermost" one only. I.e. the one not inside parentheses in your symbolic version.

3. Some Symbols [15 points]

In this question you'll do two symbolic proofs. For both, you should use propositional logic notation and rules (e.g. don't use the boolean algebra reference sheet). As in Homework 1, you should follow the symbolic proof guidelines for both parts.

(a) Prove that $(p \to q) \lor (r \to q) \equiv (p \land r) \to q$ [10 points]

Our proof has three "intermediate goals": convert to only ands/ors/nots with only atomic propositions negated, rearrange to eliminate the "extra" q, rearrange to final expression. Your proof is allowed to go differently (we will accept any correct, properly formatted proof), but our intermediate goals may help you if you are stuck.

(b) Prove that $(p \land \neg q) \equiv \neg(\neg q \to \neg p)$. [5 points]

Remember to "stay on target" – make sure you know where you're headed on this problem, and you can work from the bottom up if you're careful.

4. The New Normal Form [10 points]

Consider the following function $C(p, q, r)$:

p	q	r	$C(p, q, r)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

(a) Express C in Conjuntive Normal Form using Boolean Algebra notation. [5 points]

(b) Express C in Disjunctive Normal Form using propositional logic notation. [5 points]

5. A tale of two $\exists$ [15 points]

Consider the following two expressions:

$$\exists x(\mathtt{P}(x) \land \mathtt{Q}(x)) \qquad \exists x\, \mathtt{P}(x) \land \exists x\, \mathtt{Q}(x)$$

(a) Give a domain of discourse and definitions of P and Q such that these expressions are **not** equivalent. Explain why your examples work (1-2 sentences). [6 points]

(b) Give a domain of discourse and definitions of P and Q such that these expressions **are** equivalent. Explain why your examples work (1-2 sentences). [6 points]

(c) There is a logical relationship between these two expressions (one that is true for all domains and all predicates P, Q). By "logical relationship" we mean there is a logical connective that can join the two expressions together into a single true expression. What is that combined expression? Very briefly summarize why the relationship is true (1-2 sentences). [3 points]

6. Inside Baseball

In the beforetimes, you went to a UW baseball game with two friends on "Bark at the Park" day. Husky Baseball Stadium rules do not allow for non-human mammals to attend, except as follows: (1) Dubs is allowed at every game (2) if it is "Bark at the Park" day, everyone can bring their pet dogs. You let your domain of discourse be all mammals at the game.

The predicates Dog, Dubs, Human are true if and only if the input is a dog, Dubs, or a human respectively. UW is facing the Oregon State Beavers. The predicate $\texttt{HuskyFan}(x)$ means "x is a Husky fan" and similarly for BeaverFan. Finally HavingFun is true if and only if the input mammal is having fun right now.

6.1. Strike One [16 points]

One of your friends hands you the following observations; translate them into English. Your translations should take advantage of "restricting the domain" to make more natural translations when possible, but you should not otherwise simplify the expression before translating.

(a) $\forall x\,(\texttt{Dog}(x) \to [\texttt{Dubs}(x) \lor \texttt{BeaverFan}(x)])$

(b) $\exists x\,(\texttt{HuskyFan}(x) \land \texttt{Human}(x) \land \neg\,\texttt{HavingFun}(x))$

(c) $\forall x(\texttt{BeaverFan}(x) \to \neg\,\texttt{HavingFun}(x)) \land \forall x(\texttt{HuskyFan}(x) \lor \texttt{Dubs}(x) \to \texttt{HavingFun}(x))$

(d) $\neg\exists x\,(\texttt{Dog}(x) \land \texttt{HavingFun}(x) \land \texttt{BeaverFan}(x))$

6.2. Strike Two [8 points]

You realize that the first two sentences above are false.

(a) State the negation of (a) in English. You should simplify the negation so that the English sentence is natural.

(b) Repeat the directions above for sentence (b).

7. Extra Credit

7.1. Existential Crisis

Let's do Problem 5 again, with slightly changed expressions:

$$\exists x(\texttt{P}(x) \lor \texttt{Q}(x)) \qquad \exists x\,\texttt{P}(x) \lor \exists x\,\texttt{Q}(x)$$

If these are equivalent, give us a brief, but clear explanation of why they are. If they are not equivalent, answer all the parts from Problem 5.

7.2. XNOR

Computers have storage spaces called "registers" (they are placed right near the processing unit to hold the values urgently needed for upcoming calculations). A register is a fixed number of bits long (i.e. a fixed number of T or F). For any two bits a, b we define $\texttt{XNOR}(a, b) := \neg(a \oplus b)$.

Suppose you have two memory registers R_i and R_j. You have only one operation available: $\texttt{XNOR}(R_i, R_j)$ performs XNOR bit-by-bit and **stores the result back in** R_i. By "bit-by-bit" we mean we XNOR the k^{th} bit of R_i with the k^{th} bit of R_j to get the k^{th} bit of the result).

Show that you can swap the contents of R_i and R_j using only XNOR operations and **only** the registers R_i, R_j – you are not allowed any "temporary variables" or other registers. Give both a list of steps and a brief explanation of how your solution works.