Homework 4: English Proofs

Due date: Friday October 30 at 11:59 PM (Seattle time, i.e. GMT-7)
If you work with others (and you should!), remember to follow the collaboration policy.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting.

Be sure to read the grading guidelines for more information on what we're looking for.

This is the first homework with English proofs. Please look again at the grading guidelines now that we're doing English proofs.

1. Formal and English [23 points]

In this problem, we'll practice writing both Formal and English proofs. Let your domain of discourse be integers.
Define $\texttt{Even}(x)$ to be true if and only if $\exists k(x = 2k)$.
Define $\texttt{divBy6}(x)$ to be true if and only if $6 \mid x$.

(a) Give a predicate definition of $\texttt{divBy6}(x)$, that uses an $\exists$ quantifier. [3 points]

(b) Show that $\forall x(\texttt{divBy6}(x) \to \texttt{Even}(x))$, using an inference proof. Let the domain of discourse for your proof be integers.
You may use the definitions of predicates in the problem (including your answer to part a), as well as "algebra" to complete the proof. [9 points]

(c) Write an English proof to show the if 6 divides an integer x, then x is even. Recall that English proofs don't have domains of discourse, so you need to define types for your variables. [9 points]

(d) Go through your English proof, for each sentence in it, state which step(s) of your inference proof it most closely corresponds to (it's ok if a few steps overlap or don't correspond to a particular sentence, but this shouldn't happen to a lot of steps.). [4 points]

2. Set Proofs [24 points]

Let A, B, C be arbitrary sets. For each of the following claims: if it is true, give an English proof. If it is false, disprove it with an English proof (If you need to disprove the statement, remember that we've seen only one proof technique in class for disproving a $\forall$).

(a) $(B \setminus A) \cup (C \setminus A) = (B \cup C) \setminus A$.

(b) $A \cap \overline{(B \cap \overline{A})} = A$.

(c) $A \cap (B \cup C) = (A \cap B) \cup (B \cap C)$.

3. I've never seen such raw power[sets] [24 points]

Let S, T be arbitrary sets. For each of the following claims: if it is true, give an English proof. If it is false, disprove it with an English proof (if you need to disprove the statement, remember that we've seen only one proof technique in class for disproving a $\forall$).

(a) $\mathcal{P}(S \cup T) = \mathcal{P}(S) \cup \mathcal{P}(T)$.

(b) $\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T)$.

(c) $\mathcal{P}(S \setminus T) = \mathcal{P}(S) \setminus \mathcal{P}(T)$.

4. [C]arti[e]san-al Products [14 points]

(a) Let A, B, and C be nonempty sets. Show that if $(A \times B = A \times C)$ then $B = C$. [10 points]

(b) We required A to be nonempty in part (a), show that if we drop that requirement, the claim becomes false. [4 points]

5. Divide[s] we fall [14 points]

(a) Write an English proof showing that for any **positive** integers p, q, r if $p \mid q$ and $q \mid r$ then $p \mid r$. [8 points]

(b) Write an English proof showing that for any **positive** integers p, q if $p \mid q$ and $q \mid p$, then $p = q$.
For this problem, you may not use the result of Section 4's problem 5a as a fact, but you may find that proof useful to model yours after. [6 points]

(c) Based on these two facts, computer scientists sometimes say that divides "works kinda like $\leq$"[1] in the sense that it puts numbers "in an order." Ponder how that might work as a result of these two facts. You do not have to write anything for this part. [0 points]

6. Additive Inverses [16 points]

(a) Fix an integer a. Call an integer b an "additive inverse of $a \pmod n$" if $a + b \equiv 0 \pmod n$. Show that every integer a has an additive inverse. [8 points]

(b) Show that if there are integers a, a' such that both a and a' are additive inverses of b then $a \equiv a' \pmod n$. [8 points]

(c) Ponder why people sometimes combine the two statements above to say "Every number has a unique additive inverse." But also why this is a little different from the example of "unique" we saw in class. You do not have to write anything for this part [0 points]

[1] Ok, we have a really fancy vocabulary word for this. You'll learn it in a few weeks. For now "works kinda like $\leq$"