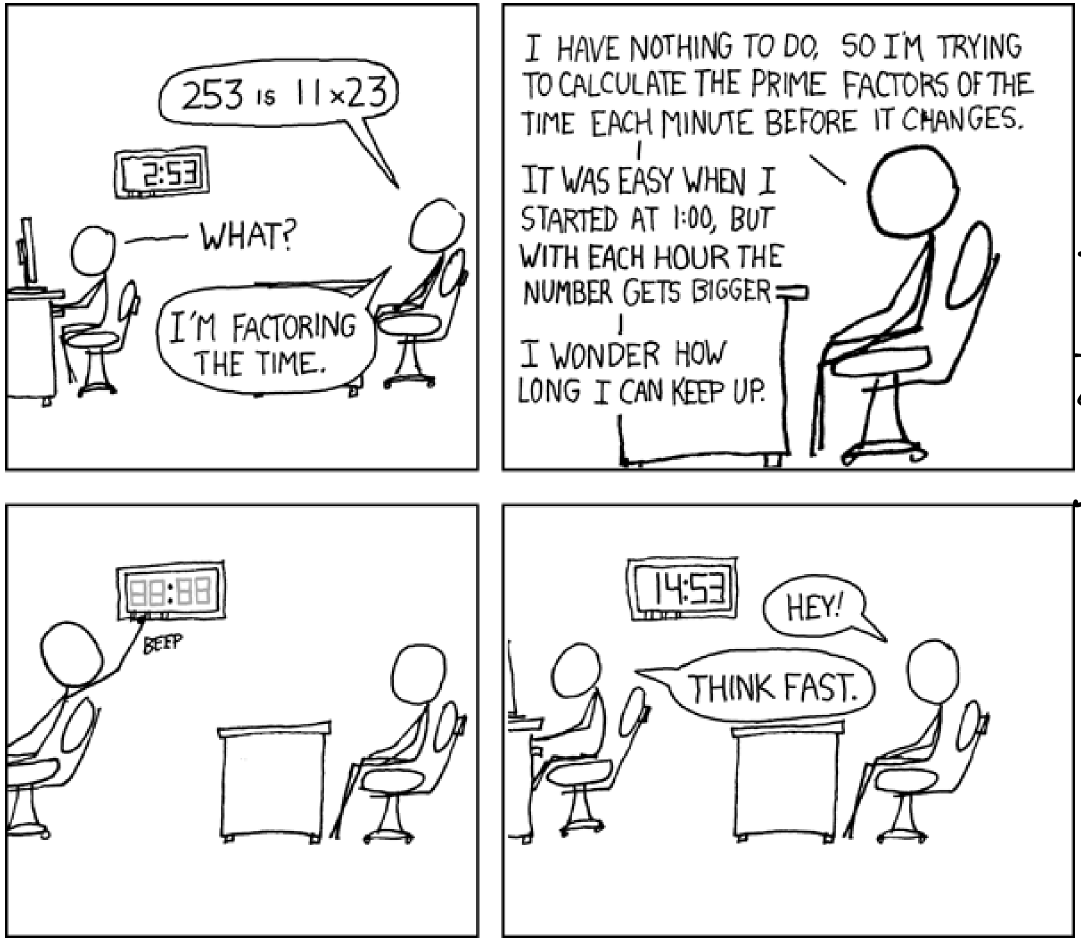
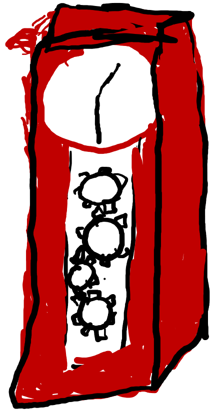


xkcd.com/247

Warm up:
Try this proof:
If $a \mid (bc)$
then $a \mid b$ or $a \mid c$.

you might get stuck.
Is there a different implication you could show?



GCD and the Euclidian Algorithm

CSE 311 Fall 2020
Lecture 13

Extra Set Practice

Show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Proof:
Firse, we'll show: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
Let x be an arbitrary element of$A \cup (B \cap C)$.
Then by definition of ∪,∩ we have:
$x \in A \vee (x \in B \wedge x \in C)$
Applying the distributive law, we get
$(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$
Applying the definition of union, we have:
$x \in (A \cup B)$ and $x \in (A \cup C)$
By definition of intersection we have $x \in (A \cup B) \cap (A \cup C)$.
So $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Now we show $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
Let x be an arbitrary element of $(A \cup B) \cap (A \cup C)$.
By definition of intersection and union, $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$
Applying the distributive law, we have $x \in A \vee (x \in B \wedge x \in C)$
Applying the definitions of union and intersection, we have $x \in A \cup (B \cap C)$
So $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
Combining the two directions, since both sets are subsets of each other, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Extra Set Practice

Suppose $A \subseteq B$. Show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Let A, B be arbitrary sets such that $A \subseteq B$.

Let X be an arbitrary element of $\mathcal{P}(A)$.

By definition of powerset, $X \subseteq A$.

Since $X \subseteq A$, every element of X is also in A. And since $A \subseteq B$, we also have that every element of X is also in B.

Thus $X \in \mathcal{P}(B)$ by definition of powerset.

Since an arbitrary element of $\mathcal{P}(A)$ is also in $\mathcal{P}(B)$, we have $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Extra Set Practice

Disprove: If $A \subseteq (B \cup C)$ then $A \subseteq B$ or $A \subseteq C$

Consider $A = \{1,2,3\}, B = \{1,2\}, C = \{3,4\}$.

$B \cup C = \{1,2,3,4\}$ so we do have $A \subseteq (B \cup C)$, but $A \nsubseteq B$ and $A \nsubseteq C$.

$3 \in A$ but
$3 \notin B$

When you disprove a $\forall$, you're just providing a counterexample (you're showing $\exists$) – your proof won't have "let x be an arbitrary element of A."

Facts about modular arithmetic

For all integers a, b, c, d, n where $n > 0$:

If $\underline{a \equiv b(mod\ n)}$ and $\underline{c \equiv d(mod\ n)}$ then $\underline{a + c \equiv b + d}(mod\ n)$.

If $\underline{a \equiv b}(mod\ n)$ and $c \equiv d(mod\ n)$ then $ac \equiv bd\ (mod\ n)$.

$a \equiv b(mod\ n)$ if and only if $b \equiv a\ (mod\ n)$.

$a\%n = (a - n)\%n$.

We didn't prove the first, it's a good exercise! You can use it as a fact as though we had proven it in class.

Another Proof

For all integers, a, b, c: Show that if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

Proof:

Let a, b, c be arbitrary integers, and suppose $a \nmid (bc)$.

$5 \nmid -1$

Then there is not an integer z such that $az = bc$

$-1 \nmid 5$

$\forall z \quad az \neq bc \quad (\text{if } z \in \mathbb{Z}).$

$-5 \nmid -5$

…

There is not an integer x such that $ax = b$, or there is not an integer y such that $ay = c$.

So $a \nmid b$ or $a \nmid c$

Another Proof

For all integers, a, b, c: Show that if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

Proof:

Let a, b, c be arbitrar … c).

Then there is not an

…

So $a \nmid b$ or $a \nmid c$

There has to be a better way!

Another Proof

For all integers, a, b, c: Show that if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

There has to be a better way!

If only there were some equivalent implication…

One where we could negate everything…

$p \to q$

$\neg q \to \neg p$

Take the contrapositive of the statement:

For all integers, a, b, c: Show if $a|b$ and $a|c$ then $a|(bc)$.

By contrapositive

Claim: For all integers, a, b, c: Show that if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

We argue by contrapositive.

Let a, b, c be arbitrary integers, and suppose $a|b$ and $a|c$.

Therefore $a|bc$

By contrapositive

Claim: For all integers, a, b, c: Show that if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

We argue by contrapositive. $\neg(p \lor q) \equiv \neg p \land \neg q$

Let a, b, c be arbitrary integers, and suppose $a|b$ and $a|c$.

By definition of divides, $ax = b$ and $ay = c$ for integers x and y.

Multiplying the two equations, we get $axay = bc$

Since a, x, y are all integers, xay is an integer. Applying the definition of divides, we have $a|bc$.

So for all integers a, b, c if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

Try it yourselves!

Show for any sets A, B, C: if $A \nsubseteq (B \cup C)$ then $A \nsubseteq C$.

1. What do the terms in the statement mean?
2. What does the statement as a whole say?
3. Where do you start?
4. What's your target?
5. Finish the proof ☺

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Or text cse311 to 22333

Try it yourselves!

$A \subseteq B$
$B \subseteq C$
$\Rightarrow A \subseteq C$

Show for any sets A, B, C: if $A \nsubseteq (B \cup C)$ then $A \nsubseteq C$.

$\forall A, B, C$: if $A \subseteq C$ then $A \subseteq (B \cup C)$

Proof:

$\exists x (x \in A \wedge x \notin B \cup C)$

We argue by contrapositive,

$\forall x (\neg x \in A \vee x \in B \cup C)$

Let A, B, C be arbitrary sets, and suppose $A \subseteq C$.

Let x be an arbitrary element of A. By definition of subset, $x \in C$. By definition of union, we also have $x \in B \cup C$. Since x was an arbitrary element of A, we have $A \subseteq (B \cup C)$.

Since A, B, C were arbitrary, we have: if $A \nsubseteq (B \cup C)$ then $A \nsubseteq C$.

Divisors and Primes

Primes and FTA

Prime

An integer $p > 1$ is prime iff its only positive divisors are 1 and p. Otherwise it is "composite"

Fundamental Theorem of Arithmetic

Every positive integer greater than 1 has a unique prime factorization.

$50 = \underbrace{2 \cdot 5^2}$

GCD and LCM

Greatest Common Divisor

The Greatest Common Divisor of a and b (gcd(a,b)) is the largest integer c such that $c|a$ and $c|b$

$\frac{6}{12}$ $\quad$ gcd(6,12) = 6 $\quad$ $\frac{1}{2}$

Least Common Multiple

The Least Common Multiple of a and b (lcm(a,b)) is the smallest positive integer c such that $a|c$ and $b|c$.

$\frac{1}{2} + \frac{2}{3} = \frac{3}{6} + \frac{4}{6} = \frac{7}{6}$

Try a few values…

gcd(100,125) = 25

gcd(17,49) = 1

gcd(17,34) = 17

gcd(13,0) = 13

$13 \overset{?}{\mid} 13$

$13 \cdot 1 = 13$ ✓

$13 \overset{?}{\mid} 0$

$13 \cdot 0 = 0$ ✓

lcm(7,11) = 77

lcm(6,10) = 30

$b \mid 13$

$b > 13$

$bz = 13$

$z \in \mathbb{Z}$

```
public int Mystery(int m, int n){
     if(m<n){
          int temp = m;
          m=n;
          n=temp;
     }
     while(n != 0) {
          int rem = m % n;
          m=n;
          n=rem;
     }
     return m;
}
```

How do you calculate a gcd?

You could:

Find the prime factorization of each

Take all the common ones. E.g.

gcd(24,20)=gcd($2^3 \cdot 3, 2^2 \cdot 5$) = 2^{min(2,3)} = 2^2 = 4.

(lcm has a similar algorithm – take the maximum number of copies of everything)

But that's....really expensive. Mystery from a few slides ago finds gcd.

Two useful facts

gcd Fact 1

If a, b are positive integers, then $\gcd(a, b) = \gcd(b, a\%b)$

Tomorrow's lecture we'll prove this fact. For now: just trust it.

gcd Fact 2

Let a be a positive integer: $\gcd(a, 0) = |a|$

$\neq 0$

Does $a|a$ and $a|0$? Yes $a \cdot 1 = a$; $a \cdot 0 = 0$.

Does anything greater than a divide a?

```
public int Mystery(int m, int n){
      if(m<n){
            int temp = m;
            m=n;
            n=temp;
      }
      while(n != 0) {
            int rem = m % n;
            m=n;
            n=rem;
      }
      return m;
}
```

$m \geq n$

$gcd(m,n)$

$= gcd(n, m\%n)$

$\vdots$

$= gcd(?, 0)$

Euclid's Algorithm

```
while(n != 0) {
    int rem = m % n;
    m=n;
    n=rem;
}
```

gcd(660,126) = gcd(126, 660 % 126) = gcd(126, 30)

= gcd(30, 126 % 30) = gcd(30, 6)

= gcd(6, 30 % 6) = gcd(6, 0)

= 6.

Tableau: m q n r

660 = 5 · 126 + 30

126 = 4 · 30 + 6

30 = 5 · 6 + 0

Euclid's Algorithm

```
while(n != 0) {
        int rem = m % n;
        m=n;
        n=rem;
    }
```

$$
\begin{aligned}
\gcd(660,126) &= \gcd(126, 660 \bmod 126) &&= \gcd(126, 30)\\
&= \gcd(30, 126 \bmod 30) &&= \gcd(30, 6)\\
&= \gcd(6, 30 \bmod 6) &&= \gcd(6, 0)\\
&= 6
\end{aligned}
$$

Tableau form

$$
\begin{aligned}
\boxed{\overset{m}{660}} &= 5 \cdot \boxed{\overset{n}{126}} + 30\\
126 &= 4 \cdot 30 + 6\\
30 &= 5 \cdot 6 + 0
\end{aligned}
$$

Starting Numbers

Final answer

Bézout's Theorem

Bézout's Theorem

If a and b are positive integers, then there exist integers s and t such that

$$\gcd(a,b) = sa + tb$$

$\gcd(-a,-b) = \gcd(a,b) = sa + tb$

$= -s\,a - t\,b$

We're not going to prove this theorem…

But we'll show you how to find s,t for any positive integers a, b.

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

gcd(35,27)

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

gcd(35,27) = gcd(27, 35%27) = gcd(27,8)
= gcd(8, 27%8) = gcd(8, 3)
= gcd(3, 8%3) = gcd(3, 2)
= gcd(2, 3%2) = gcd(2,1)
= gcd(1, 2%1) = gcd(1,0)

m q n r

$$\boxed{\begin{aligned} 35 &= 1 \cdot 27 + 8 \\ 27 &= 3 \cdot 8 + 3 \\ 8 &= 2 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \end{aligned}}$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

m q n r

$$
\boxed{\begin{array}{l}
35 = 1 \cdot 27 + 8 \\
27 = 3 \cdot 8 + 3 \\
8 = 2 \cdot 3 + 2 \\
3 = 1 \cdot 2 + 1
\end{array}}
$$

r m q n

$8 = 35 - 1 \cdot 27$

$3 = 27 - 3 \cdot 8$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$\boxed{\begin{array}{l} 35 = 1 \cdot 27 + 8 \\ 27 = 3 \cdot 8 + 3 \\ 8 = 2 \cdot 3 + 2 \\ 3 = 1 \cdot 2 + 1 \end{array}}$$

r m q n

$$\boxed{\begin{array}{l} 8 = 35 - 1 \cdot 27 \\ 3 = 27 - 3 \cdot 8 \\ 2 = 8 - 2 \cdot 3 \\ 1 = 3 - 1 \cdot 2 \end{array}}$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

m		n
$8 = 35$	$- 1 \cdot$	27
$3 = 27$	$- 3 \cdot$	8
$2 = 8$	$- 2 \cdot$	3
$1 = 3$	$- 1 \cdot$	2

gcd(m, n)

$\gcd(m,n) = s\,m + t\,n$

$3 \cdot \boxed{3} - 3 \cdot 8$

$-(y - 2x)$

$-y + 2x$

$g(35,27) = 1 = 3 - 1 \cdot 2$

$= 3 - 1 \cdot (8 - 2 \cdot 3)$

$= 3 + -1 \cdot 8 + 2 \cdot 3$

$= 3 \cdot 3 - 1 \cdot 8$

$= 3(27 - 3 \cdot 8) + -1 \cdot 8$

$= 3 \cdot 27 + -10 \cdot 8$

$= 3 \cdot 27 + -10(35 - 1 \cdot 27)$

$= 13 \cdot 27 + -10 \cdot 35$

$1 = 3 - 1 \cdot 2$

$= 3 - 1(8 - 2 \cdot 3)$

$= 3 - 1 \cdot 8 + 2 \cdot 3$

$= 3 \cdot 3 + -1 \cdot 8$

$= 3(27 - 3 \cdot 8) + -1 \cdot 8$

$= 3 \cdot 27 - 9 \cdot 8 + -1 \cdot 8$

$= 3 \cdot 27 - 10 \cdot 8$

$3 \cdot 27 - 10(35 - 1 \cdot 27)$

$(3 + 10)27 - 10 \cdot 35$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$\boxed{\begin{aligned} 8 &= 35 - 1 \cdot 27 \\ 3 &= 27 - 3 \cdot 8 \\ 2 &= 8 - 2 \cdot 3 \\ 1 &= 3 - 1 \cdot 2 \end{aligned}}$$

$$\begin{aligned} 1 &= 3 - 1 \cdot 2 \\ &= 3 - 1 \cdot (8 - 2 \cdot 3) \\ &= -1 \cdot 8 + 2 \cdot 3 \end{aligned}$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$\boxed{\begin{array}{l} 8 = 35 - 1 \cdot 27 \\ 3 = 27 - 3 \cdot 8 \\ 2 = 8 - 2 \cdot 3 \\ 1 = 3 - 1 \cdot 2 \end{array}}$$

$$\begin{aligned} 1 &= 3 - 1 \cdot 2 \\ &= 3 - 1 \cdot (8 - 2 \cdot 3) \\ &= -1 \cdot 8 + 3 \cdot 3 \\ &= -1 \cdot 8 + 3(27 - 3 \cdot 8) \\ &= 3 \cdot 27 - 10 \cdot 8 \\ &= 3 \cdot 27 - 10(35 - 1 \cdot 27) \\ &= 13 \cdot 27 - 10 \cdot 35 \end{aligned}$$

When substituting back, you keep the larger of m, n and the number you just substituted. Don't simplify further! (or you lose the form you need)

gcd(27,35) = $13 \cdot 27 + (-10) \cdot 35$

So…what's it good for?

Suppose I want to solve $7x \equiv 1(mod\ n)$

Just multiply both sides by $\frac{1}{7}$…

Oh wait. We want a number to multiply by 7 to get 1.

If the gcd(7,n) = 1

Then $s \cdot 7 + tn = 1$, so $7s - 1 = -tn$ i.e. $n|(7s-1)$ so $7s \equiv 1(mod\ n)$.

So the s from Bézout's Theorem is what we should multiply by!

Try it

Solve the equation $7y \equiv 3(mod\ 26)$

What do we need to find?

The multiplicative inverse of $7(\mathbf{mod}\ 26)$

Finding the inverse…

gcd(26,7) = gcd(7, 26%7) = gcd(7,5)

= gcd(5, 7%5) = gcd(5,2)

= gcd(2, 5%2) = gcd(2, 1)

= gcd(1, 2%1) = gcd(1,0)= 1.

$$26 = 3 \cdot 7 + 5\,;\ 5 = 26 - 3 \cdot 7$$

$$7 = 5 \cdot 1 + 2\,;\ 2 = 7 - 5 \cdot 1$$

$$5 = 2 \cdot 2 + 1\,;\ 1 = 5 - 2 \cdot 2$$

$$1 = 5 - 2 \cdot 2$$
$$= 5 - 2(7 - 5 \cdot 1)$$
$$= 3 \cdot 5 - 2 \cdot 7$$
$$= 3 \cdot (26 - 3 \cdot 7) - 2 \cdot 7$$
$$3 \cdot 26 - 11 \cdot 7$$

-11 is a multiplicative inverse.
We'll write it as 15, since we're working mod 26.

Try it

Solve the equation $7y \equiv 3(mod\ 26)$

What do we need to find?
The multiplicative inverse of $7\ (mod\ 26)$.

$15 \cdot 7 \cdot y \equiv 15 \cdot 3(mod\ 26)$

$y \equiv 45(mod\ 26)$

Or $y \equiv 19(mod\ 26)$

So $26|19 - y$, i.e. $26k = 19 - y$ (for $k \in \mathbb{Z}$) i.e. $y = 19 - 26 \cdot k$ for any $k \in \mathbb{Z}$

So $\{\dots, -7, 19, 45, \dots 19 + 26k, \dots\}$

And now, for some proofs!

GCD fact

If a and b are positive integers, then gcd(a,b) = gcd(b, a % b)

How do you show two gcds are equal?

Call $a = \gcd(w, x), b = \gcd(y, z)$

If $b|w$ and $b|x$ then b is a common divisor of w, x so $b \leq a$

If $a|y$ and $a|z$ then a is a common divisor of $y, z,$ so $a \leq b$

If $a \leq b$ and $b \leq a$ then $a = b$

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that y is a common divisor of a and b.

By definition of gcd, $y|b$ and $y|(a\%b)$. So it is enough to show that $y|a$.

Applying the definition of divides we get $b = yk$ for an integer k, and $(a\%b) = yj$ for an integer j.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q.

Plugging in both of our other equations:

$a = qyk + yj = y(qk + j)$. Since q, k, and j are integers, $y|a$. Thus y is a common divisor of a, b and thus $y \leq x$.

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

By definition of gcd, $\mathrm{x}|b$ and $x|a$. So it is enough to show that $\mathrm{x}|(a\%b)$.

Applying the definition of divides we get $b = xk'$ for an integer k', and $\mathrm{a} = xj'$ for an integer j'.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q

Plugging in both of our other equations:

$xj' = qxk' + a\%b$. Solving for $a\%b$, we have $a\%b = xj' - qxk' = x(j' - qk')$. So $x|(a\%b)$. Thus x is a common divisor of $b, a\%b$ and thus $x \leq y$.

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

We have shown $x \leq y$ and $y \leq x$.

Thus $x = y$, and $\gcd(a, b) = \gcd(b, a\%b)$.