Midterm Misconceptions

Induction Problem

Call a line “properly ordered” if it meets all of those requirements.
Show that for all $n \geq 2$, in every properly ordered line there are two people wearing gold hats next to each other.

$P(n)$: “Every properly ordered line with n pairs has two consecutive people wearing gold hats.”

Base Case: $n=2$ P G G P

We need to show a $\forall$ statement in the inductive step.

To prove a for all statement, the first thing we do in our proof is…

Introduce an arbitrary variable!

Induction Problem

So if you didn't start with "let L be an arbitrary properly ordered line with $k+1$ pairs of people" you didn't start in the right place.

If you started with "an arbitrary properly ordered line with k pairs"

There's not formally a way to argue that "by listing out all the possible alterations I could think of, I'll end up with all the possible lines of length $k+1$"

You might have (you probably did) but it's still not a rigorous argument of a forall statement if you don't start with an arbitrary line of length $k+1$.

Induction Problem

L = PGGP

PG GG GP

This kind of attempted induction argument (where you "build up" to a supposedly arbitrary element from an arbitrary smaller element) easily hides bugs. For that reason it's not logically valid.

See: HW6 P6.

Never ever ever try to prove a "for all" induction by building up (ever).

Always start with the arbitrary big thing (the $k+1$ thing) and find the smaller thing inside.

There is no rule of inference that says "I started with an arbitrary thing and did some alterations and it's now an arbitrary other thing"

Induction Problem

But wait…don’t we just do that when we prove inequalities by induction?

Nope!

1. Inequalities aren’t for-all statements (or if they are you introduce the variable at the start, like we did for that string induction proof)

2. We prove inequalities the normal way we prove inequalities (either starting from a fact you know and deriving the desired inequality, or starting from the left hand side and altering it until you get the right hand side).

Induction Problem

But wait, don't we "build up" when we do structural induction?

Nope!

Basis:

Recursive:

Exclusion: nothing else is in S

The recursive definition in structural induction guarantees us what the arbitrary element looks like…it's made up of two 'smaller' elements in the set.

…and the template just lets us skip the words "let T be arbitrary, by the recursive definition, T is of the form…"

Induction Problem

But wait, that stamp collecting problem. We definitely started with the small one there.

The stamp collecting induction was an exists statement (there is a way to build $k + 1$). So yeah, we definitely didn't have anything arbitrary there.

Nor would we expect to – it was an exists statement!

Set Problems

Notes from the TAs

Be careful with set-builder notation

Using variables you've defined in spots where dummy variables are expected:

1. Does not mean what you think it means.
2. "Hurts [your TA's] brain"

Dummy Variables

A lot like a local variable in Java.

It means something only inside its method.

$\int 5x^2dx$ x is a dummy variable. It means something inside the integral, (so you can write dx) but wouldn't necessarily mean anything outside.

$\exists x(P(x) \wedge Q(x))$ x is a dummy variable.

$\{y : y^2 \geq 5\}$ y is a dummy variable

Dummy Variables

So if you said something like

Let y be an arbitrary element of output,

Consider $\{y : y = x\}$ this y is not that y

Set Proofs

If you're showing $A \subseteq B$

Your first step should **always** be

Let x be an arbitrary element of A.

A lot of you had attempted proofs where you tried to write $output(f, A \cap B) = \{y : \exists x\ f(x) = y\}$ and modify the inside

$\{y : \exists x \ \dots\}$

Don't do this. It's never how you do a set proof.

Set problems

This was a hard problem.

It's what we call a "synthesis" problem – applying familiar ideas and techniques in new combinations.

You should expect these types of problems in all of your future courses if you haven't seen them already; the end goal of university education is synthesis.

You needed to combine set proofs, set builder notation, quantifier notation (both exists and for-all) to do this problem.

When a problem is hard, it's easy to get overwhelmed.

Take a deep breath, and do the 4-step process.

1. What do the words in the statement mean?
2. What does the statement as a whole mean?
3. Where do I start?
4. Where is my target?

"What types are these objects?"

For any function f and any set S, define output$(f, S) = \{y : \exists x(x \in S \land f(x) = y)\}$

For example, output$(x^2, \{1, 2, 3\}) = \{y : \exists x(x \in \{1, 2, 3\} \land f(x) = y)\} = \{1, 4, 9\}$, and output$(x^2, \{-1, 1\}) = \{1\}$.

(a) Show that output$(f, A \cap B) \subseteq$ output$(f, A) \cap$ output(f, B). [10 points]

Let f be an arbitrary function, let A, B be arbitrary sets.

Let y be an arbitrary element of output$(f, A \cap B)$

Therefore $y \in$ output$(f, A) \cap$ output(f, B).

For any function f and any set S, define output$(f, S) = \{y : \exists x(x \in S \wedge f(x) = y)\}$

For example, output$(x^2, \{1,2,3\}) = \{y : \exists x(x \in \{1,2,3\} \wedge f(x) = y)\} = \{1,4,9\}$, and output$(x^2, \{-1,1\}) = \{1\}$.

(a) Show that output$(f, A \cap B) \subseteq$ output$(f, A) \cap$ output(f, B). [10 points]

Let f be an arbitrary function, let A, B be arbitrary sets.

Let y be an arbitrary element of output$(f, A \cap B)$

By defn of output, ~~there~~ is an $x \in A \cap B$ s.t. $f(x) = y$

By defn of intersection $x \in A$ and $x \in B$

Since $x \in A$ and $f(x) = y$, $y \in$ output(f, A)

" $x \in B$ " " $y \in$ output(f, B)

Thus

$y \in$ output$(f, A) \wedge y \in$ output(f, B)

So $y \in$ output$(f, A) \cap$ output(f, B)

For any function f and any set S, define output$(f, S) = \{y : \exists x(x \in S \wedge f(x) = y)\}$

For example, output$(x^2, \{1,2,3\}) = \{y : \exists x(x \in \{1,2,3\} \wedge f(x) = y)\} = \{1,4,9\}$, and output$(x^2, \{-1,1\}) = \{1\}$.

(a) Show that output$(f, A \cap B) \subseteq$ output$(f, A) \cap$ output(f, B). [10 points]

Let f be an arbitrary function, let A, B be arbitrary sets.

Let y be an arbitrary element of output$(f, A \cap B)$

By definition of output, there is an x such that $x \in A \cap B$ and $f(x) = y$

$y \in$output(f, A) and $y \in$output(f, B)

So $y \in$output$(f, A) \cap$output(f, B)

For any function f and any set S, define output$(f, S) = \{y : \exists x(x \in S \land f(x) = y)\}$

For example, output$(x^2, \{1, 2, 3\}) = \{y : \exists x(x \in \{1, 2, 3\} \land f(x) = y)\} = \{1, 4, 9\}$, and output$(x^2, \{-1, 1\}) = \{1\}$.

(a) Show that output$(f, A \cap B) \subseteq$ output$(f, A) \cap$ output(f, B). [10 points]

Let f be an arbitrary function, let A, B be arbitrary sets.

Let y be an arbitrary element of output$(f, A \cap B)$

By definition of output, there is an x such that $x \in A \cap B$ and $f(x) = y$

Since $x \in A \cap B$ $x \in A$ and $x \in B$.

So by definition of output, $f(x) = y \in$output(f, A) and $f(x) = y \in$output(f, B)

$y \in$output(f, A) and $y \in$output(f, B)

So $y \in$output$(f, A) \cap$output(f, B)

(d) "non-repetitive" $\forall x,y\ (f(x)=f(y) \rightarrow x=y)$

if f is non-repetitive, $\boxed{\text{output}(f,A) \cap \text{output}(f,B)} \subseteq \boxed{\text{output}(f, A\cap B)}$

Proof: Let A, B be arbitrary sets,
Suppose f is an arbitrary non-repetitive function.
Let y be an arbitrary element of output(f, A) ∩ output(f, B).
By defn of ∩, y ∈ output(f, A) and y ∈ output(f, B).
By defn of output there is an $x \in A$ s.t. $f(x)=y$; there is an x' $x' \in B$, $f(x')=y$
Because $f(x)=f(x')$, we can apply defn of non-repetitive and get $x=x'$
$x \in A$ and $x \in B$, so $x \in A \cap B$, and $f(x)=y$

$\boxed{\exists x\ x \in A\cap B,\ f(x)=y}$

$\therefore y \in \text{output}(f, A\cap B)$

Translation

Here's a process:

1. Read through in order, looking for any quantifiers.
2. Write the "core" assertion
3. Add in any domain restriction.

$\forall x \exists y (Book(x) \rightarrow$

$\wedge Person(y))$

"Hank Green has written exactly two books"

$$\exists x \exists y \big(\mathrm{Author}(\mathrm{HankGreen}, x) \wedge \mathrm{Author}(\mathrm{HankGreen}, y) \wedge x \neq y \wedge \forall z \, [\mathrm{Author}(\mathrm{HankGreen}, z) \to (z = x \vee z = y)] \big)$$

x is the unique x st. $P(x)$

$$\forall z \, (P(z) \to z = x)$$