Section 08: Solutions

1. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$f(0) = 0$$
$$f(1) = 1$$
$$f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

Solution:

Let $P(n)$ be "$f(n) = n$". We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction on n.

Base Cases $(n = 0, n = 1)$**:** $f(0) = 0$ and $f(1) = 1$ by definition.

Inductive Hypothesis: Assume that $P(0) \wedge P(1) \wedge \ldots P(k)$ hold for some arbitrary $k \geq 1$.

Inductive Step: We show $P(k+1)$:

$$\begin{aligned} f(k+1) &= 2f(k) - f(k-1) && \text{[Definition of } f\text{]} \\ &= 2(k) - (k-1) && \text{[Induction Hypothesis]} \\ &= k+1 && \text{[Algebra]} \end{aligned}$$

Conclusion: $P(n)$ is true for all $n \in \mathbb{N}$ by principle of strong induction.

2. Walk the Dawgs

Suppose a dog walker takes care of $n \geq 12$ dogs. The dog walker is not a strong person, and will walk dogs in groups of 4 or 5 at a time (every dog gets walked exactly once). Prove the dog walker can always split the n dogs into groups of 4 or 5.

Solution:

Let $P(n)$ be "a group with n dogs can be split into groups of 4 or 5 dogs." We will prove $P(n)$ for all natural numbers $n \geq 12$ by strong induction.

Base Cases $n = 12, 13, 14,$ **or** 15**:** 12 = 4 + 4 + 4, 13 = 4 + 4 + 5, 14 = 4 + 5 + 5, 15 = 5 + 5 + 5. So $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.

Inductive Hypothesis: Assume that $P(12), \ldots, P(k)$ hold for some arbitrary $k \geq 15$.

Inductive Step: Goal: Show $k+1$ dogs can be split into groups of size 4 or 5.

We first form one group of 4 dogs. Then we can divide the remaining $k-3$ dogs into groups of 4 or 5 by the assumption $P(k-3)$. (Note that $k \geq 15$ and so $k-3 \geq 12$; thus, $P(k-3)$ is among our assumptions $P(12), \ldots, P(k)$.)

Conclusion: $P(n)$ holds for all integers $n \geq 12$ by by principle of strong induction.

3. Reversing a Binary Tree

Consider the following definition of a (binary) **Tree**.

Basis Step `Nil` is a **Tree**.

Recursive Step If L is a **Tree**, R is a **Tree**, and x is an integer, then $\texttt{Tree}(x, L, R)$ is a **Tree**.

The sum function returns the sum of all elements in a **Tree**.

$$\begin{aligned} \mathsf{sum}(\texttt{Nil}) &= 0 \\ \mathsf{sum}(\texttt{Tree}(x, L, R)) &= x + \mathsf{sum}(L) + \mathsf{sum}(R) \end{aligned}$$

The following recursively defined function produces the mirror image of a **Tree**.

$$\begin{aligned} \mathsf{reverse}(\texttt{Nil}) &= \texttt{Nil} \\ \mathsf{reverse}(\texttt{Tree}(x, L, R)) &= \texttt{Tree}(x, \mathsf{reverse}(R), \mathsf{reverse}(L)) \end{aligned}$$

Show that, for all **Tree**s T that

$$\mathsf{sum}(T) = \mathsf{sum}(\mathsf{reverse}(T))$$

Solution:

For a **Tree** T, let $P(T)$ be "$\mathsf{sum}(T) = \mathsf{sum}(\mathsf{reverse}(T))$". We show $P(T)$ for all **Tree**s T by structural induction.

Base Case: By definition we have $\mathsf{reverse}(\texttt{Nil}) = \texttt{Nil}$. Applying sum to both sides we get $\mathsf{sum}(\texttt{Nil}) = \mathsf{sum}(\mathsf{reverse}(\texttt{Nil}))$, which is exactly $P(\texttt{Nil})$, so the base case holds.

Inductive Hypothesis: Suppose $P(L)$ and $P(R)$ hold for some arbitrary **Tree**s L and R.

Inductive Step: Let x be an arbitrary integer. Goal: Show $P(\texttt{Tree}(x, L, R))$ holds.

We have,

$$\begin{aligned} \mathsf{sum}(\mathsf{reverse}(\texttt{Tree}(x, L, R))) &= \mathsf{sum}(\texttt{Tree}(x, \mathsf{reverse}(R), \mathsf{reverse}(L))) && \text{[Definition of reverse]} \\ &= x + \mathsf{sum}(\mathsf{reverse}(L)) + \mathsf{sum}(\mathsf{reverse}(L)) && \text{[Definition of sum]} \\ &= x + \mathsf{sum}(R) + \mathsf{sum}(L) && \text{[Inductive Hypothesis]} \\ &= x + \mathsf{sum}(L) + \mathsf{sum}(R) && \text{[Commutativity]} \\ &= \mathsf{sum}(\texttt{Tree}(x, L, R)) && \text{[Definition of sum]} \end{aligned}$$

This shows $P(\texttt{Tree}(x, L, R))$.

Conclusion: Therefore, $P(T)$ holds for all **Tree**s T by structural induction.

4. Bernoulli's Inequality

Show that for any integer $n \geq 0$ and real number $x \geq -1$ that $(1 + x)^n \geq 1 + nx$.

Solution:

Let $P(n)$ be "for any real number $x \geq -1$ it holds that $(1 + x)^n \geq 1 + nx$." We show $P(n)$ for all integer $n \geq 0$ by induction on n.

Base Case: For any real number $x \geq -1$ we have $(1 + x)^0 = 1 = 1 + 0(x)$ (Note, we assume $0^0 = 1$ here). Thus, $P(0)$ holds.

Inductive Hypothesis: Suppose $P(k)$ holds for some arbitrary integer $k \geq 0$.

Industive Step: Goal: For any real number $x \geq -1$, $(1+x)^{k+1} \geq 1+(k+1)x$.

Let x be an arbitrary real number such that $x \geq -1$. Then,

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)(1+x)^k \\
&\geq (1+x)(1+kx) && \text{[IH, and since } 1+x \geq 0\text{]} \\
&= 1+kx+x+kx^2 && \text{[Distribute terms]} \\
&= 1+(k+1)x+kx^2 && \text{[Factor out } x\text{]} \\
&\geq 1+(k+1)x && \text{[Since } kx^2 \geq 0\text{]}
\end{aligned}$$

Thus, $(1+x)^{k+1} \geq 1+(k+1)x$, so $P(k+1)$ holds.

Conclusion: $P(n)$ holds for all integers $n \geq 0$ by the principle of induction.

5. Regular Expressions

(a) Write a regular expression that matches base 10 numbers (e.g., there should be no leading zeroes).

Solution:

$$0 \cup ((1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)(0 \cup 1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)^*)$$

(b) Write a regular expression that matches all base-3 numbers that are divisible by 3.

Solution:

$$0 \cup ((1 \cup 2)(0 \cup 1 \cup 2)^* 0)$$

(c) Write a regular expression that matches all binary strings that contain the substring "111", but not the substring "000".

Solution:

$$(01 \cup 001 \cup 1^*)^*(0 \cup 00 \cup \varepsilon)111(01 \cup 001 \cup 1^*)^*(0 \cup 00 \cup \varepsilon)$$