Section 08: Induction, Regular Expressions

1. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$f(0) = 0$$
$$f(1) = 1$$
$$f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

2. Walk the Dawgs

Suppose a dog walker takes care of $n \geq 12$ dogs. The dog walker is not a strong person, and will walk dogs in groups of 4 or 5 at a time (every dog gets walked exactly once). Prove the dog walker can always split the n dogs into groups of 4 or 5.

3. Reversing a Binary Tree

Consider the following definition of a (binary) **Tree**.

Basis Step `Nil` is a **Tree**.

Recursive Step If L is a **Tree**, R is a **Tree**, and x is an integer, then $\texttt{Tree}(x, L, R)$ is a **Tree**.

The sum function returns the sum of all elements in a **Tree**.

$$\begin{aligned} \mathsf{sum}(\texttt{Nil}) &= 0 \\ \mathsf{sum}(\texttt{Tree}(x, L, R)) &= x + \mathsf{sum}(L) + \mathsf{sum}(R) \end{aligned}$$

The following recursively defined function produces the mirror image of a **Tree**.

$$\begin{aligned} \mathsf{reverse}(\texttt{Nil}) &= \texttt{Nil} \\ \mathsf{reverse}(\texttt{Tree}(x, L, R)) &= \texttt{Tree}(x, \mathsf{reverse}(R), \mathsf{reverse}(L)) \end{aligned}$$

Show that, for all **Tree**s T that

$$\mathsf{sum}(T) = \mathsf{sum}(\mathsf{reverse}(T))$$

4. Bernoulli's Inequality

Show that for any integer $n \geq 0$ and real number $x \geq -1$ that $(1+x)^n \geq 1 + nx$.

5. Regular Expressions

(a) Write a regular expression that matches base 10 numbers (e.g., there should be no leading zeroes).

(b) Write a regular expression that matches all base-3 numbers that are divisible by 3.

(c) Write a regular expression that matches all binary strings that contain the substring “111”, but not the substring “000”.