Section 04: Propositions and Proofs

1. Formal Spoofs

For each of the following proofs, determine why the proof is incorrect. Then, consider whether the conclusion of the proof is true or not. If it is true, state how the proof could be fixed. If it is false, give a counterexample.

(a) Show that $\exists z \, \forall x \; P(x, z)$ follows from $\forall x \, \exists y \; P(x, y)$.

1.	$\forall x \, \exists y \; P(x, y)$	Given
2.	$\forall x \; P(x, c)$	$\exists$ Elim: 1 (c special)
3.	$\exists z \, \forall x P(x, z)$	$\exists$ Intro: 2

(b) Show that $\exists z \; (P(z) \wedge Q(z))$ follows from $\forall x \; P(x)$ and $\exists y \; Q(y)$.

1.	$\forall x \; P(x)$	Given
2.	$\exists y \; Q(y)$	Given
3.	Let z be arbitrary	
4.	$P(z)$	Elim $\forall$: 1
5.	$Q(z)$	Elim $\exists$: 2 (z special)
6.	$P(z) \wedge Q(z)$	Intro $\wedge$: 4, 5
7.	$\exists z \; P(z) \wedge Q(z)$	Intro $\exists$: 6

2. Predicate Logic Formal Proof

Given $\forall x \, T(x) \to M(x)$, we wish to prove $(\exists x \, T(x)) \to (\exists y \, M(y))$.

The following formal proof does this, but it is missing explanations for each line. Fill in the blanks with inference rules or equivalences to apply (as well as the line numbers) to complete the proof.

1. $\forall x \, T(x) \to M(x)$ (____________________)
    - 2.1. $\exists x \, T(x)$ (____________________)
    - 2.2. $T(c)$ (____________________)
    - 2.3. $T(c) \to M(c)$ (____________________)
    - 2.4. $M(c)$ (____________________)
    - 2.5. $\exists y \, M(y)$ (____________________)
2. $(\exists x \, T(x)) \to (\exists y \, M(y))$ (____________________)

3. A Formal Proof in Predicate Logic

Prove $\exists x \; (P(x) \vee R(x))$ from $\forall x \; (P(x) \vee Q(x))$ and $\forall y \; (\neg Q(y) \vee R(y))$.

4. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

(b) $B = \{\{\},\ \{\{\}\},\ \{\{\},\{\}\},\ \{\{\},\{\},\{\}\},\ \ldots\}$

(c) $D = \varnothing$

(d) $E = \{\varnothing\}$

(e) $C = A \times (B \cup \{7\})$

5. Game, Set, Match

Prove each of the following set identities.

(a) $A \setminus B \subseteq A \cup C$ for any sets A, B, C. (Formally and then in English.)

(b) $(A \setminus B) \setminus C \subseteq A \setminus C$ for any sets A, B. (Formally and then in English)

(c) $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D. (English only.)

6. Ghosts and Skeletons

Let A and B be sets and P and Q be predicates. For each of the claims below, write the *skeleton* of an English proof of the claim. It will not be possible to complete the proof with just the information given, but you should be able to see the basic shape of the proof.

For example, suppose we want to prove "No element of A satisfies P." Then, our proof would have this shape:

> Let x be arbitrary.
> Suppose that $x \in A$. **....** Thus, $P(x)$ is false.
> Since x was arbitrary, this shows that no element of A satisfies P.

This shows the general shape (skeleton) of the proof. We don't know how to complete the proof since we don't know what A and P are. For any particular choice of A and P, though, the proof would still look like this but with the "...." replaced by specific reasoning for that A and P.

Note that we have actually proven $\forall x\, \neg P(x)$, whereas the claim best translates as $\neg\exists x\, P(x)$. However, the two are equivalent by De Morgan's law, and that is a simple enough step that the reader should see it.

(a) $A = B$

(b) Any object that satisfies P but not Q is in the set B.

(c) B is not a subset of A.