Homework 6: Induction

Due date: Wednesday May 12 at 11:00 PM (Seattle time, i.e. GMT-7)
If you work with others (and you should!), remember to follow the collaboration policy.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting. Whenever an exercise asks to "*prove*" or "*show*" a statement without further specification, you are free to choose any proof method that you have learned in 311.

Be sure to read the grading guidelines for more information on what we're looking for.

While working on the assignment, please (roughly) keep track of the time you spent on each problem. This is so that you have an idea of how to answer the feedback questions!

1. Collaborators

List any collaborators (i.e., other students) you worked with and which problems you worked on together, or state that you worked alone.

2. Induction and sums [10 points]

Prove that, for every positive integer n, the following equality is true

$$6 \cdot \sum_{k=1}^{n} k^2 = n(2n^2 + 3n + 1)$$

3. Induction Divides [20 points]

Prove that $7 \mid (8^n - 1)$ for all $n \in \mathbb{N}$, by induction on n.

Hint: In your inductive step, you'll need to be creative to apply your inductive hypothesis. Focus on forcing the right expression to appear.

4. Induction Code [20 points]

Consider the following code snippet.

```
public int Mystery(int n){
    if(n < 0)
        throw new InvalidInputException();
    if(n == 0)
        return 2;
    if(n == 1)
        return 7;
    return Mystery(n-1) + 2*Mystery(n-2);
}
```

Use strong induction to show that `Mystery(n)`$= 3 \cdot 2^n + (-1)^{n+7}$ for all integers $n \geq 0$.

5. Well that just doesn't sound right [8 points]

Consider the following (very incorrect) induction proof:

① Let $P(n)$ be "$5n = 0$"
We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

② Base Case: $n = 0$
If $n = 0$ then $5n = 5 \cdot 0 = 0$, so $P(0)$ is true.

③ Inductive Hypothesis: Suppose $P(n)$ holds for $n = 0, \ldots, k$ for an arbitrary integer $k \geq 0$

④ Inductive Step:
Ⓐ We want to prove $P(k+1)$ is true, i.e. $5(k+1) = 0$.
Ⓑ Observe that $5(k+1) = 5(s) + 5(t)$. for integers s, t with $0 \leq s < k+1$ and $0 \leq t < k+1$.
Ⓒ Applying the inductive hypothesis twice, we have $5s = 0$ and $5t = 0$.
Ⓓ Substituting both into the original equation, we get: $5(k+1) = 0 + 0$, so $5(k+1) = 0$, as required.
⑤ The result follows for all $n \geq 0$ by induction.

(a) Find the smallest counterexample to the claim that $P(n)$ holds for all $n \in \mathbb{N}$. [3 points] You should both (1) show that your example is a counterexample and (2) argue why all smaller natural numbers are not counterexamples.

(b) Clearly identify the flaw in the proof; it will help to run through the proof with your smallest counterexample. For ease of explanation, we've taken the (unusual) step of labelling every sentence. [5 points]

6. Running Times [20 points]

You wrote a piece of recursive code. On an input of size n, your function takes $T(n)$ time to run, where:

$$\begin{aligned} T(n) &= 5n && \text{if } 1 \leq n \leq 4 \\ T(n) &= T(\lfloor n/2 \rfloor) + T(\lfloor n/4 \rfloor) + 5n && \text{for all } n > 4 \end{aligned}$$

In the definition above, $\lfloor x \rfloor$ is the "floor" function; it returns the greatest integer less than or equal to x.
For example: $\lfloor 3.2 \rfloor = 3$, $\lfloor 3.7 \rfloor = 3$, $\lfloor 3 \rfloor = 3$.

Show that for all $n \in \mathbb{N}$ with $n \geq 1$, $T(n) \leq 20n$

Hint 1: Notice that while $T()$ is defined with equality, you are only proving an inequality.
Hint 2: The only fact about the floor function you will need is $\lfloor x \rfloor$ is an integer and $x - 1 < \lfloor x \rfloor \leq x$.

7. str000ng induction would be a good choice [20 points]

Let 0^n mean a string of n zeros. Let S be the set of strings defined as follows:

Basis Steps: $0^3 \in S$, $0^5 \in S$
Recursive Step: If $0^x, 0^y \in S$ then $0^x \cdot 0^y \in S$ where $\cdot$ is string concatenation.

Show that, for every integer $n \geq 12$ the set S contains the string 0^n.

Caution: Structural Induction is not the best tool for this problem. Structural induction shows $\forall x. x \in S \rightarrow P(x)$. You're analyzing what the elements of S are in this problem, not proving a predicate holds for all elements of S.

8. Extra Credit: Sticks And Stones

Consider an infinite sequence of positions $1, 2, 3, \ldots$ and suppose we have a stone at position 1 and another stone at position 2. In each step, we choose one of the stones and move it according to the following rule: Say we decide to move the stone at position i; if the other stone is not at any of the positions $i+1, i+2, \ldots, 2i$, then it goes to $2i$, otherwise it goes to $2i+1$.

For example, in the first step, if we move the stone at position 1, it will go to 3 and if we move the stone at position 2 it will go to 4. Note that, no matter how we move the stones, they will never be at the same position.

Use induction to prove that, for any given positive integer n, it is possible to move one of the stones to position n. For example, if $n = 7$ first we move the stone at position 1 to 3. Then, we move the stone at position 2 to 5 Finally, we move the stone at position 3 to 7.

9. Feedback

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment?
- Which problem did you spend the most time on?
- Which problem did you find to be the most confusing?
- Any other feedback for us?