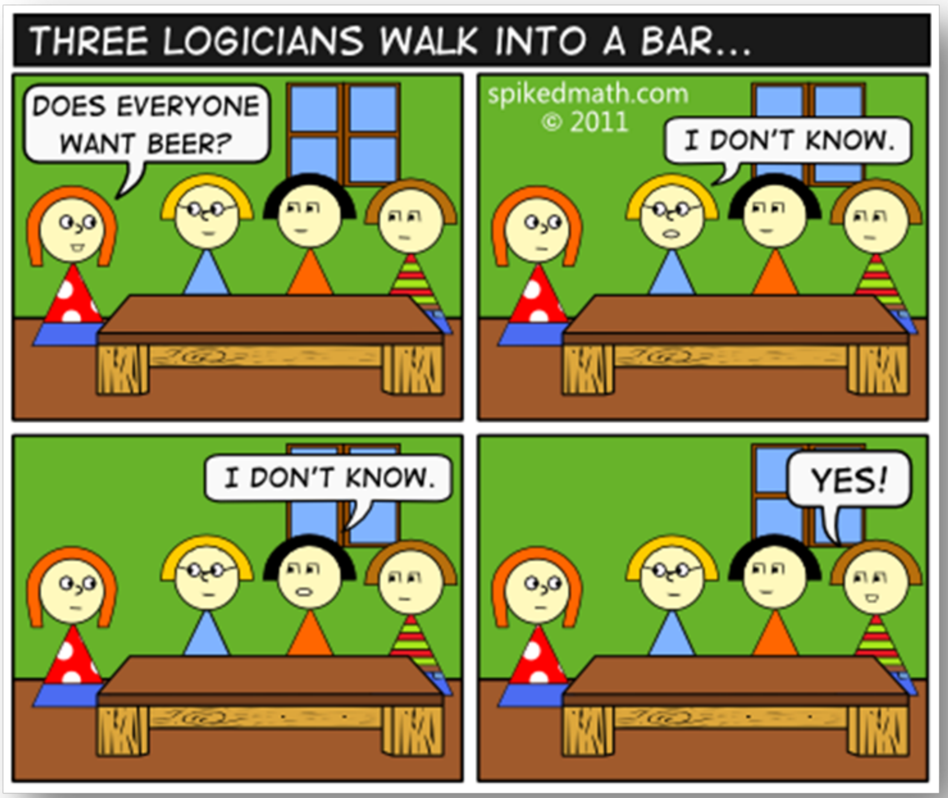
CSE 311: Foundations of Computing

Lecture 6: More Predicate Logic



Asking homework questions on Ed

- **New system on Ed: we are creating threads of the form "Homework X, Problem Y" for each homework problem.**
- **If you have a question for a particular homework problem, please:**
  - **Read the whole thread**
  - **Then ask the question in that thread**

- Link: https://edstem.org/us/courses/4896/discussion/

Last class: Intro to predicate logic

- **Domain of discourse** = variable range
- **Predicates**: Functions $P(x)$ that return a truth value for each x in domain (predicates may depend on more than one variable, e.g. $Q(x_1, x_2, x_3)$)
- **Existential quantor** ∃ and **universal quantor** ∀

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Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	

Determine the truth values of each of these statements:

∀x (Even(x) ∨ Odd(x))

∃x (Even(x) ∧ Odd(x))

∀x Greater(x+1, x)

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Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
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Determine the truth values of each of these statements:

∀x (Even(x) ∨ Odd(x)) **T** every integer is either even or odd

∃x (Even(x) ∧ Odd(x)) **F** no integer is both even and odd

∀x Greater(x+1, x) **T** adding 1 makes a bigger number

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

"Red cats like tofu"

"Some red cats don't like tofu"

English to Predicate Logic

Domain of Discourse
Mammals

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Cat(x) ::= "x is a cat"
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"Red cats like tofu"

$$\forall x\, ((\text{Red}(x) \land \text{Cat}(x)) \rightarrow \text{LikesTofu}(x))$$

"Some red cats don't like tofu"

$$\exists y\, ((\text{Red}(y) \land \text{Cat}(y)) \land \neg \text{LikesTofu}(y))$$

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
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LikesTofu(x) ::= "x likes tofu"

When putting two predicates together like this, we use an "and".

"Red cats like tofu"

When restricting to a smaller domain in a "for all" we use **implication.**

When there's no leading quantification, it means "for all".

"Some red cats don't like tofu"

When restricting to a smaller domain in an "exists" we use **and.**

"Some" means "there exists".

Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= "x is a purple fruit"

(*) $\forall$x PurpleFruit(x) ("All fruits are purple")

What is the negation of (*)?

(a) "there exists a purple fruit"

(b) "there exists a non-purple fruit"

(c) "all fruits are not purple"

Try your intuition! Which one "feels" right?

Key Idea: In every domain, exactly one of a statement and its negation should be true.

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Domain of Discourse
{plum}

(*), (a)

Domain of Discourse
{apple}

(b), (c)

Domain of Discourse
{plum, apple}

(a), (b)

Negations of Quantifiers

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Key Idea: In every domain, exactly one of a statement and its negation should be true.

Domain of Discourse
{plum}

(*), (a)

Domain of Discourse
{apple}

(b), (c)

Domain of Discourse
{plum, apple}

(a), (b)

The only choice that ensures exactly one of the statement and its negation is (b).

De Morgan’s Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

"There is no largest integer"

$$\neg \exists x \forall y\ (x \geq y)$$
$$\equiv \forall x \neg \forall y\ (x \geq y)$$
$$\equiv \forall x\ \exists y \neg (x \geq y)$$
$$\equiv \forall x\ \exists y\ (y > x)$$

"For every integer, there is a larger integer"

Scope of Quantifiers

$\exists x \; (P(x) \wedge Q(x))$ **vs.** $\exists x \, P(x) \wedge \exists x \, Q(x)$

scope of quantifiers

$\exists x\ (P(x) \wedge Q(x))$ **vs.** $\exists x\ P(x) \wedge \exists x\ Q(x)$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

Scope of Quantifiers

Example: NotLargest(x) $\equiv \exists$ y Greater (y, x)
$\equiv \exists$ z Greater (z, x)

truth value:

doesn't depend on y **or** z **"bound variables"**

does depend on x **"free variable"**

quantifiers only act on free variables of the formula they quantify

$$\forall x\ (\exists y\ (P(x,y) \rightarrow \forall x\ Q(y, x)))$$

Quantifier "Style"

$$\forall x(\exists y\,(P(x,y) \rightarrow \forall\, x\, Q(y, x)))$$

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…

Nested Quantifiers

- **Bound variable names don't matter**

$$\forall x\ \exists y\ P(x, y) \equiv \forall a\ \exists b\ P(a, b)$$

- **Positions of quantifiers can sometimes change**

$$\forall x\ (Q(x) \wedge \exists y\ P(x, y)) \equiv \forall x\ \exists y\ (Q(x) \wedge P(x, y))$$

- **But: order is important...**

Quantifier Order Can Matter

Domain of Discourse

Integers

OR

{1, 2, 3, 4}

Predicate Definitions

GreaterEq(x, y) ::= “x ≥ y”

“There is a number greater than or equal to all numbers.”

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)))$$

“Every number has a number greater than or equal to it.”

$$\forall y\ \exists x\ \text{GreaterEq}(x, y)))$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

The purple statement requires **an entire row** to be true.

The red statement requires one entry in **each column** to be true.

Quantification with Two Variables

expression	when true	when false
$\forall x \forall y P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \exists y P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \exists y P(x, y)$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \forall x P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.

Lecture 6 Activity

- You will be assigned to **breakout rooms**. Please:
- Introduce yourself
- Choose someone to share screen, showing this PDF
- Today's task: Consider the predicate logic expression

$$\neg\exists x\ [(\forall y\ P(x, y)) \vee (\exists z\ Q(x, z))]$$

- Obtain an equivalent logic expression where negations are directly in front of the predicates.

Then fill out the poll everywhere for **Activity Credit**!
Go to pollev.com/thomas311 and login with your UW identity

De Morgan Laws:

$$\neg\forall x\ P(x) \equiv \exists x \neg P(x)$$
$$\neg\exists x\ P(x) \equiv \forall x \neg P(x)$$

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If p and p $\rightarrow$ q are both true then q must be true

- Write this rule as $$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

- Given:
  - If it is Friday, then you have a 311 class today.
  - It is Friday.

- Therefore, by Modus Ponens:
  - You have a 311 class today.

Inference Rules

If A is true and B is true ....

Requirements: $\dfrac{A \;;\; B}{}$

Conclusions: $\therefore C \;,\; D$

Then, C must be true

Then D must be true

Example (Modus Ponens):

$$\frac{A \;;\; A \rightarrow B}{\therefore \quad B}$$

If I have A and $A \rightarrow B$ both true,
Then B must be true.

Axioms: Special inference rules

If I have nothing…

Requirements: $\overline{}$

Conclusions: $\therefore C \;,\; D$

Then, C must be true

Then D must be true

Example (Excluded Middle):

$$\frac{}{\therefore A \vee \neg A}$$

$A \vee \neg A$ must be true.

My First Proof!

Show that s follows from q, $q \rightarrow r$, and $r \rightarrow s$

1.	q	Given
2.	$q \rightarrow r$	Given
3.	$r \rightarrow s$	Given
4.		
5.		

My First Proof!

Show that s follows from q, $q \rightarrow r$, and $r \rightarrow s$

1.	q	Given
2.	$q \rightarrow r$	Given
3.	$r \rightarrow s$	Given
4.	r	MP: 1, 2
5.	s	MP: 3, 4

Proofs can use equivalences too

Show that $\neg q$ follows from $q \rightarrow r$ and $\neg r$

1.	$q \rightarrow r$	Given
2.	$\neg r$	Given
3.		
4.		

Proofs can use equivalences too

Show that $\neg q$ follows from $q \rightarrow r$ and $\neg r$

1.	$q \rightarrow r$	Given
2.	$\neg r$	Given
3.	$\neg r \rightarrow \neg q$	Contrapositive: 1
4.		

Proofs can use equivalences too

Show that $\neg q$ follows from $q \rightarrow r$ and $\neg r$

1.	$q \rightarrow r$	Given
2.	$\neg r$	Given
3.	$\neg r \rightarrow \neg q$	Contrapositive: 1
4.	$\neg q$	MP: 2, 3

Simple Propositional Inference Rules

Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it

$$\frac{q \land r}{\therefore q, r} \qquad \frac{q, r}{\therefore q \land r}$$

$$\frac{q \lor r, \neg q}{\therefore r} \qquad \frac{q}{\therefore q \lor r}$$

$$\frac{q, q \rightarrow r}{\therefore r} \qquad \frac{p \Rightarrow q}{\therefore p \rightarrow q}$$

Direct Proof Rule

Not like other rules

Proofs

Show that r follows from p, p → q and (p ∧ q) → r

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

$$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

$$\frac{p \wedge q}{\therefore\ p,\ q}$$

$$\frac{p,\ q}{\therefore\ p \wedge q}$$

Proofs

Show that r follows from $p, p \rightarrow q$, and $p \wedge q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.		
4.		
5.		
6.		

Proofs

Show that r follows from $p, p \rightarrow q$, and $p \land q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	q	MP: 1, 2
4.		
5.		
6.		

Proofs

Show that r follows from $p, p \to q$, and $p \land q \to r$

1.	p	Given
2.	$p \to q$	Given
3.	q	MP: 1, 2
4.	$p \land q$	Intro $\land$: 1, 3
5.		
6.		

Proofs

Show that r follows from $p, p \rightarrow q,$ and $p \wedge q \rightarrow r$

1. p Given
2. $p \rightarrow q$ Given
3. q MP: 1, 2
4. $p \wedge q$ Intro $\wedge$: 1, 3
5. $p \wedge q \rightarrow r$ Given
6.

Proofs

Show that r follows from $p, p \rightarrow q,$ and $p \wedge q \rightarrow r$

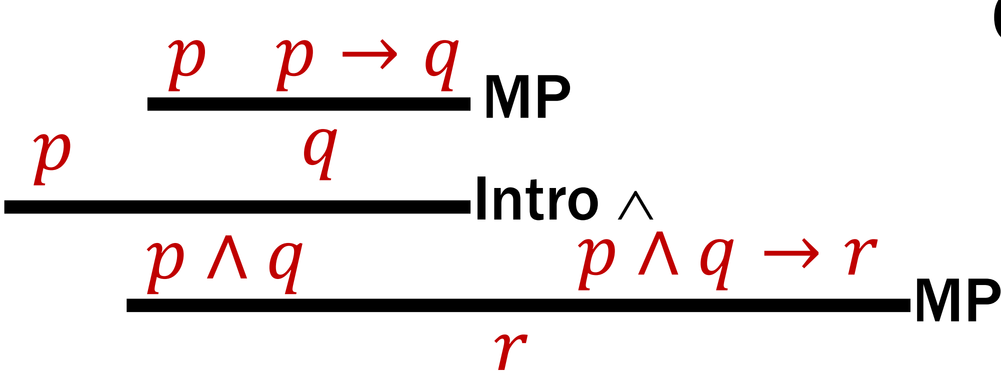
1.	p	Given
2.	$p \rightarrow q$	Given
3.	q	MP: 1, 2
4.	$p \wedge q$	Intro $\wedge$: 1, 3
5.	$p \wedge q \rightarrow r$	Given
6.	r	MP: 4, 5

Proofs

Show that r follows from $p, p \to q,$ and $p \land q \to r$

1.	p	Given
2.	$p \to q$	Given
3.	q	MP: 1, 2
4.	$p \land q$	Intro $\land$: 1, 3
5.	$p \land q \to r$	Given
6.	r	MP: 4, 5

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that's great!



Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.
- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \to q$ given

~~2. $(p \vee r) \to q$ intro $\vee$ from 1.~~

Does not follow! e.g . p=F, q=F, r=T

Proofs

Prove that $\neg r$ follows from $p \land s$, $q \rightarrow \neg r$, and $\neg s \lor q$.

1. $p \land s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \lor q$ Given

First: Write down givens and goal

20. $\neg r$?

Idea: Work backwards!

Proofs

Prove that $\neg r$ follows from $p \land s$, $q \rightarrow \neg r$, and $\neg s \lor q$.

1.	$p \land s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \lor q$	Given
20.	$\neg r$	MP: 2, ?

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- We can use $q \rightarrow \neg r$ to get there.
- The justification between 2 and 20 looks like "elim $\rightarrow$" which is MP.

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
19.	q	?
20.	$\neg r$	MP: 2, 19

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- Now, we have a new "hole"
- We need to prove q...
  - Notice that at this point, if we prove q, we've proven $\neg r$...

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
19.	q	?
20.	$\neg r$	MP: 2, 19

This looks like or-elimination.

Elim ∨ $\dfrac{A \vee B \,;\, \neg A}{\therefore B}$

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
18.	$\neg\neg s$	?
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

$\neg\neg s$ doesn't show up in the givens but s does and we can use equivalences

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
17.	s	?
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

Proofs

Prove that $\neg r$ follows from $p \land s$, $q \to \neg r$, and $\neg s \lor q$.

1.	$p \land s$	Given
2.	$q \to \neg r$	Given
3.	$\neg s \lor q$	Given
17.	s	$\land$ Elim: 1
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\lor$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

No holes left! We just need to clean up a bit.

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given
4. s $\wedge$ Elim: 1
5. $\neg\neg s$ Double Negation: 4
6. q $\vee$ Elim: 3, 5
7. $\neg r$ MP: 2, 6