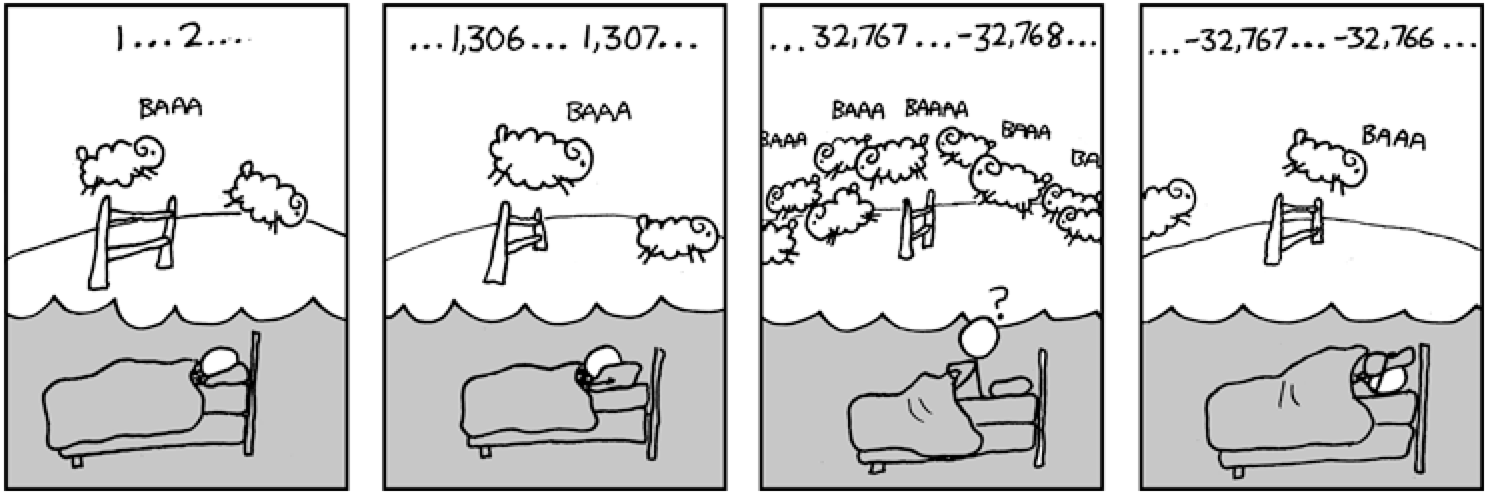
CSE 311: Foundations of Computing

Lecture 15: The Euclidean algorithm and applications



Recap from last lecture

- An integer $p > 2$ is prime if the only positive factors are 1 and p
- A prime factorization for an integer $n > 1$ is of the form $n = p_1 \cdot \ldots \cdot p_k$ where $p_1, \ldots, p_k$ are prime numbers.
- Each integer $n > 1$ as a unique prime factorization.
- The greatest common divisor $\gcd(a, b)$ is the largest integer d with $d \mid a$ and $d \mid b$.
- Important fact for today: $\gcd(a, b) = \gcd(b, a \,\%\, b)$

Another simple GCD fact

If a is a positive integer, gcd(a,0) = a.

Euclid's Algorithm

gcd(a, b) = gcd(b, a mod b), gcd(a,0)=a

```
int gcd(int a, int b){ /* a >= b, b >= 0 */
    if (b == 0) {
        return a;
    }
    else {
        return gcd(b, a % b);
    }
```

Example: GCD(660, 126)

Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

gcd(660,126) =

Euclid’s Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \,\%\, b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

$$
\begin{aligned}
\gcd(660,126) &= \gcd(126, 660 \,\%\, 126) &&= \gcd(126, 30)\\
&= \gcd(30, 126 \,\%\, 30) &&= \gcd(30, 6)\\
&= \gcd(6, 30 \,\%\, 6) &&= \gcd(6, 0)\\
&= 6
\end{aligned}
$$

Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

gcd(660,126) = gcd(126, 660 % 126) = gcd(126, 30)
= gcd(30, 126 % 30) = gcd(30, 6)
= gcd(6, 30 % 6) = gcd(6, 0)
= 6

In tableau form:

660 = 5 * 126 + 30
126 = 4 * 30 + 6
30 = 5 * 6 + 0

Bézout's theorem

If a and b are positive integers, then there exist integers $\boldsymbol{s}$ and $\boldsymbol{t}$ such that

$$\gcd(a,b) = \boldsymbol{s}a + \boldsymbol{t}b.$$

Proof is algorithmic: via the Extended Euclidean algorithm

Extended Euclidean algorithm

- **Method:** $\text{extGCD}(a, b) \to (g, s, t)$
- **Input: Integers** $a \geq b \geq 0$
- **Output: Integers** g, s, t **s.t.** $g = \gcd(a, b) = s \cdot a + t \cdot b$

1. **IF** $b == 0$ **THEN return** $(a, 1,0)$ $\quad // \ a = \gcd(a, 0) = 1 \cdot a + 0 \cdot 0$
2. $(g, s, t) \coloneqq \text{extGCD}(b, a \,\%\, b)$
3. **Write** $a = qb + (a \,\%\, b)$ **with** $q \in \mathbb{Z}$
4. **Return** $(g, t, s - tq)$

$// \ g = \gcd(b, a \,\%\, b) = \gcd(a, b)$

$// \ g = s \cdot b + t \cdot (a \,\%\, b)$

$// \ \ = s \cdot b + t \cdot (a - qb)$

$// \ \ = t \cdot a + (s - tq) \cdot b$

Extended Euclidean algorithm

- **Example: Find** s, t **such that** $\gcd(31,16) = s \cdot 31 + t \cdot 16$

Extended Euclidean algorithm

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extGCD(31, 16) -> extGCD(16, 31 % 16)

Extended Euclidean algorithm

- **Example: Find** s, t **such that** $\gcd(31,16) = s \cdot 31 + t \cdot 16$

extGCD(31, 16) -> extGCD(16, 31 % 16)

extGCD(16, 15) -> extGCD(15, 16 % 15)

Extended Euclidean algorithm

- **Example: Find s, t such that $\gcd(31,16) = s \cdot 31 + t \cdot 16$**

extGCD(31, 16) -> extGCD(16, 31 % 16)
  extGCD(16, 15) -> extGCD(15, 16 % 15)
    extGCD(15, 1) -> extGCD(1, 15 % 1)

Extended Euclidean algorithm

- **Example: Find s, t such that $\gcd(31,16) = s \cdot 31 + t \cdot 16$**

extGCD(31, 16) -> extGCD(16, 31 % 16)
extGCD(16, 15) -> extGCD(15, 16 % 15)
extGCD(15, 1) -> extGCD(1, 15 % 1)
extGCD(1, 0) = 1 = 1*1 + 0*0

Hence $\gcd(31,16) = 1$

Extended Euclidean algorithm

- **Example: Find s, t such that $\gcd(31,16) = s \cdot 31 + t \cdot 16$**

exttGCD(31, 16) -> extGCD(16, 31 % 16)

extGCD(16, 15) -> extGCD(15, 16 % 15)

extGCD(15, 1) -> extGCD(1, 15 % 1)

extGCD(1, 0) = 1 = 1*1 + 0*0

15=15*1 + 0 q s t

1 = 0*15 + (1 - 0*15)*1 = 0*15 + 1*1

Hence $\gcd(31,16) = 1$

Extended Euclidean algorithm

- **Example: Find s, t such that $\gcd(31,16) = s \cdot 31 + t \cdot 16$**

extGCD(31, 16) -> extGCD(16, 31 % 16)

extGCD(16, 15) -> extGCD(15, 16 % 15)

extGCD(15, 1) -> extGCD(1, 15 % 1)

extGCD(1, 0) = 1 = 1*1 + 0*0

s t

15=15*1 + 0 q

1 = 0*15 + (1 - 0*15)*1 = 0*15 + 1*1

16=1*15 + 1

1 = 1*16 + (0 - 1*1)*15 = 1*16 + -1*15

Hence $\gcd(31,16) = 1$

Extended Euclidean algorithm

- **Example: Find s, t such that $\gcd(31,16) = s \cdot 31 + t \cdot 16$**

extGCD(31, 16) -> extGCD(16, 31 % 16)

extGCD(16, 15) -> extGCD(15, 16 % 15)

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15=15*1 + 0

1 = 0*15 + (1 - 0*15)*1 = 0*15 + 1*1

16=1*15 + 1

1 = 1*16 + (0 - 1*1)*15 = 1*16 + -1*15

31=1*16 + 15

1 = -1*31 + (1 - -1*1)*16 = -1*31 + 2*16

Hence $\gcd(31,16) = 1 = (-1) \cdot 31 + 2 \cdot 16$

Extended Euclidean algorithm

- **Example: Find s, t such that $\gcd(31,16) = s \cdot 31 + t \cdot 16$**

extGCD(31, 16) -> extGCD(16, 31 % 16)

extGCD(16, 15) -> extGCD(15, 16 % 15)

extGCD(15, 1) -> extGCD(1, 15 % 1)

extGCD(1, 0) = 1 = **1***1 + **0***0

15=15*1 + 0

1 = **0***15 + **(1 - 0*15)***1 = **0***15 + **1***1

16=1*15 + 1

1 = **1***16 + **(0 - 1*1)***15 = **1***16 + -**1***15

31=1*16 + 15

1 = **-1***31 + **(1 - -1*1)***16 = **-1***31 + **2***16

Hence $\gcd(31,16) = 1 = (-1) \cdot 31 + 2 \cdot 16$

Multiplicative inverse $\mathrm{mod}\ m$

Suppose $\mathrm{GCD}(a, m) = 1$

By Bézout's Theorem, there exist integers s and t such that $sa + tm = 1$.

$s \% m$ is the multiplicative inverse of a:

$$1 = (sa + tm) \% m = sa \% m$$

Example

Solve: $7x \equiv 1 \pmod{26}$

Example

Solve: $7x \equiv 1 \pmod{26}$

$$\gcd(26,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,1) = 1$$

$$\begin{aligned} 26 &= 7*3 + 5 & \qquad 5 &= 26 - 7*3 \\ 7 &= 5*1 + 2 & \qquad 2 &= 7 - 5*1 \\ 5 &= 2*2 + 1 & \qquad 1 &= 5 - 2*2 \end{aligned}$$

$$\begin{aligned} 1 &= 5 - 2*(7 - 5*1) \\ &= (-7)*2 + 3*5 \\ &= (-7)*2 + 3*(26 - 7*3) \\ &= (-11)*7 + 3*26 \end{aligned}$$

Multiplicative inverse of 7 mod 26

Now $(-11)\ \overset{\%}{\cancel{\text{mod}}}\ 26 = 15$. **So,** $x = 15 + 26k$ **for** $k \in \mathbb{Z}$.

Example of a more general equation

Now solve: $7y \equiv 3 \pmod{26}$

We already computed that 15 is the multiplicative inverse of 7 modulo 26:

That is, $7 \cdot 15 \equiv 1 \pmod{26}$

By the multiplicative property of mod we have

$$7 \cdot 15 \cdot 3 \equiv 3 \pmod{26}$$

So any $y \equiv 15 \cdot 3 \pmod{26}$ is a solution.

That is, $y = 19 + 26k$ for any integer k is a solution.

Lecture 15 Activity

You will be assigned to breakout rooms. Please:

- Introduce yourself
- Choose someone to share their screen, showing this PDF
- Discuss the following questions:

1. If you run the extended Euclidean algorithm for $(51,23)$ it will return that $\gcd(51,23) = 1 = (-9) \cdot 51 + 20 \cdot 23$. What is this telling you about the multiplicative inverse of 51 modulo 23.
2. If you run the extended Euclidean algorithm for $(51,24)$ it will return that $\gcd(51,24) = 3 = 1 \cdot 51 + (-2) \cdot 24$. What is this telling you about the multiplicative inverse of 51 modulo 24.
3. What is the set of integers that do not have a multiplicative inverse modulo 10?

Fill out the poll everywhere for Activity Credit!
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Lecture 15 Activity

1. If you run the extended Euclidean algorithm for $(51, 23)$ it will return that $\gcd(51, 23) = 1 = (-9) \cdot 51 + 20 \cdot 23$. What is this telling you about the multiplicative inverse of 51 modulo 23.
   Solution: $-9 + 23 = 14$
2. If you run the extended Euclidean algorithm for $(51, 24)$ it will return that $\gcd(51, 24) = 3 = 1 \cdot 51 + (-2) \cdot 24$. What is this telling you about the multiplicative inverse of 51 modulo 24.
   Solution: There is none.
3. What is the set of integers that do not have a multiplicative inverse modulo 10?
   Solution: $0 + 10x, 2 + 20x, 5 + 10x,\ x \in \mathbb{Z}$

Fill out the poll everywhere for Activity Credit!
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Math mod a prime is especially nice

$\gcd(a, m) = 1$ if m is prime and $0 < a < m$ so can always solve these equations mod a prime.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

X	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

mod 7

Modular Exponentiation % 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3						
4						
5						
6						

Exponentiation

- **Compute** 78365^{81453}

- **Compute** $78365^{81453} \% 104729$

- **Output is small**
  - need to keep intermediate results small

Repeated Squaring – small and fast

Since $a \,\%\, m \equiv a \pmod{m}$ and $b \,\%\, m \equiv b \pmod{m}$

we have $ab \,\%\, m = \big((a \,\%\, m)(b \,\%\, m)\big) \% \, m$

So $\quad a^2 \,\%\, m \; = \; (a \,\%\, m)^2 \,\%\, m$

and $\quad a^4 \,\%\, m \; = \; (a^2 \,\%\, m)^2 \,\%\, m$

and $\quad a^8 \,\%\, m \; = \; (a^4 \,\%\, m)^2 \,\%\, m$

and $\quad a^{16} \,\%\, m \; = \; (a^8 \,\%\, m)^2 \,\%\, m$

and $\quad a^{32} \,\%\, m \; = \; (a^{16} \,\%\, m)^2 \,\%\, m$

Can compute $a^k \,\%\, m$ for $k = 2^i$ in only i steps

What if k is not a power of 2?

Fast Exponentiation Algorithm

81453 in binary is 10011111000101101

$81453 = 2^{16} + 2^{13} + 2^{12} + 2^{11} + 2^{10} + 2^{9} + 2^{5} + 2^{3} + 2^{2} + 2^{0}$

$a^{81453} = a^{2^{16}} \cdot a^{2^{13}} \cdot a^{2^{12}} \cdot a^{2^{11}} \cdot a^{2^{10}} \cdot a^{2^{9}} \cdot a^{2^{5}} \cdot a^{2^{3}} \cdot a^{2^{2}} \cdot a^{2^{0}}$

$$
\begin{aligned}
&a^{81453} \,\%\, m = \\
&(\ldots(((((a^{2^{16}} \,\%\, m \cdot \\
&\quad a^{2^{13}} \,\%\, m\,) \,\%\, m \cdot \\
&\quad a^{2^{12}} \,\%\, m) \,\%\, m \cdot \\
&\quad a^{2^{11}} \,\%\, m) \,\%\, m \cdot \\
&\quad a^{2^{10}} \,\%\, m) \,\%\, m \cdot \\
&\quad a^{2^{9}} \,\%\, m) \,\%\, m \cdot \\
&\quad a^{2^{5}} \,\%\, m) \,\%\, m \cdot \\
&\quad a^{2^{3}} \,\%\, m) \,\%\, m \cdot \\
&\quad a^{2^{2}} \,\%\, m) \,\%\, m \cdot \\
&\quad a^{2^{0}} \,\%\, m) \;\; \%\, m
\end{aligned}
$$

The fast exponentiation algorithm computes $a^k \,\%\, m$ using $\leq 2\log k$ multiplications $\%\, m$

Fast Exponentiation: $a^k \,\%\, m$ for all k

Another way....

$$a^{2j} \% \, m = \left(a^j \,\%\, m\right)^2 \% \, m$$

$$a^{2j+1} \% \, m = \left((a \,\%\, m) \cdot (a^{2j} \,\%\, m)\right) \% \, m$$

Fast Exponentiation

```
public static long FastModExp(long a, long k, long modulus) {
       long result = 1;
       long temp;

       if (k > 0) {
           if ((k % 2) == 0) {
               temp = FastModExp(a,k/2,modulus);
               result = (temp * temp) % modulus;
           }
           else {
               temp = FastModExp(a,k-1,modulus);
               result = (a * temp) % modulus;
           }
       }
       return result;
   }
```

$$a^{2j} \% \, m = \left(a^{j} \% \, m\right)^{2} \% \, m$$

$$a^{2j+1} \% \, m = \left((a \% \, m) \cdot (a^{2j} \% \, m)\right) \% \, m$$

Using Fast Modular Exponentiation

- Your e-commerce web transactions use SSL (Secure Socket Layer) based on RSA encryption
- RSA
  - Vendor chooses random 512-bit or 1024-bit primes p, q and 512/1024-bit exponent e. Computes $m = p \cdot q$
  - Vendor broadcasts (m, e)
  - To send a to vendor, you compute $C = a^e \,\%\, m$ using *fast modular exponentiation* and send C to the vendor.
  - Using secret p, q the vendor computes d that is the *multiplicative inverse* of e mod $(p-1)(q-1)$.
  - Vendor computes $C^d \,\%\, m$ using *fast modular exponentiation*.
  - **Fact:** $a = C^d \,\%\, m$ for $0 < a < m$ unless $p|a$ or $q|a$