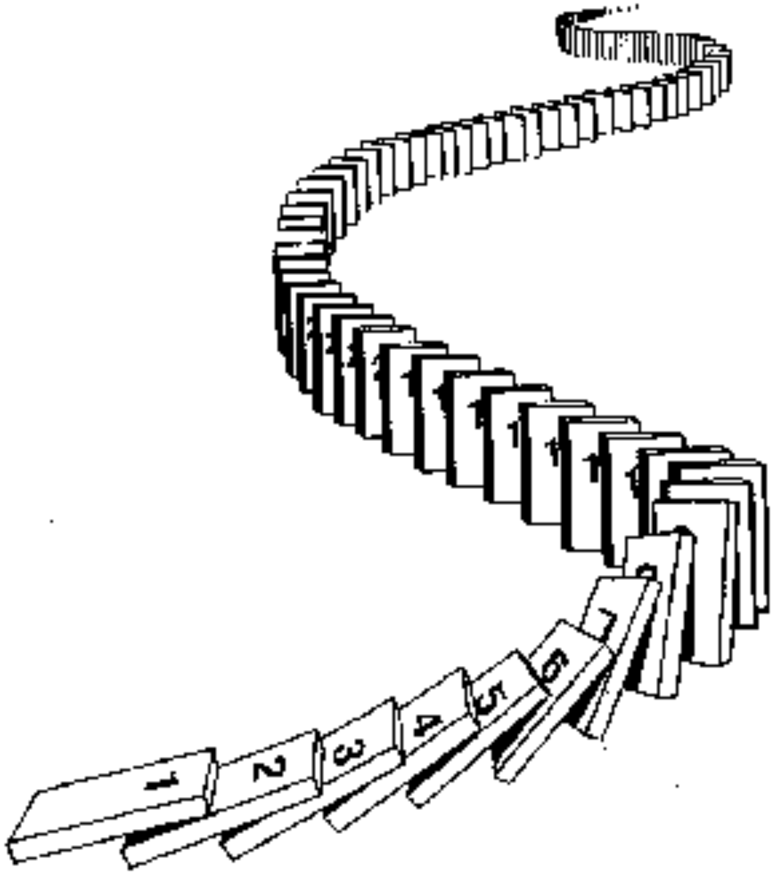
CSE 311: Foundations of Computing

Lecture 18: Recursive definitions



Recap: *Strong* Inductive Proofs

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by ***strong*** induction."
2. "Base Case:" Prove $P(b), \ldots, P(c)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq c$,

   $P(j)$ ***is true for every integer*** j ***from*** b **to** k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   You may apply the I.H. ($P(b), \ldots, P(k)$ are true) anywhere.
   Point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Recursive Definition of Sets

Recursive Definition

- Basis Step: $0 \in S$
- Recursive Step: If $x \in S$, then $x + 2 \in S$
- Exclusion Rule: Every element in S follows from basis steps and a finite number of recursive steps.

Recursive Definitions of Sets

Natural numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+1 \in S$

Even numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+2 \in S$

Powers of 3:

Basis: $1 \in S$

Recursive: If $x \in S$, then $3x \in S$.

Basis: $[0, 0] \in S$, $[1, 1] \in S$

Recursive: If $[n-1, x] \in S$ and $[n, y] \in S$, then $[n+1, x + y] \in S$. ?

Recursive Definitions of Sets

Natural numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+1 \in S$

Even numbers

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Basis: $[0, 0] \in S$, $[1, 1] \in S$

Recursive: If $[n-1, x] \in S$ and $[n, y] \in S$, then $[n+1, x + y] \in S$.

Fibonacci numbers

Recursive Definitions of Sets: General Form

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from basis steps and a finite number of recursive steps

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma$ (ε is the empty string)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes

Palindromes are strings that are the same backwards and forwards

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$

All Binary Strings with no 1’s before 0’s

All Binary Strings with no 1's before 0's

Basis:

$\mathcal{E} \in S$

Recursive:

If $\mathrm{x} \in S$**, then** $0\mathrm{x} \in S$

If $\mathrm{x} \in S$**, then** $\mathrm{x}1 \in S$

Function Definitions on Recursively Defined Sets

Length:

$\text{len}(\mathcal{E}) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\mathcal{E}^R = \mathcal{E}$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \mathcal{E} = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$

Lecture 18 Activity

- You will be assigned to **breakout rooms**. Please:
- Introduce yourself
- Choose someone to share screen, showing this PDF
- Consider the set S that is recursively defined by

Basis: $6 \in S$, $15 \in S$
Recursive: If $x, y \in S$ **then** $x + y \in S$

- List explicitly the elements of S

Fill out a poll everywhere for Activity Credit!
Go to pollev.com/thomas311 and login
with your UW identity

Recall: Fundamental Theorem of Arithmetic

Every integer > 1 has a unique prime factorization

48 = 2 • 2 • 2 • 2 • 3
591 = 3 • 197
45,523 = 45,523
321,950 = 2 • 5 • 5 • 47 • 137
1,234,567,890 = 2 • 3 • 3 • 5 • 3,607 • 3,803

We use strong induction to prove that a factorization into primes exists, but not that it is unique.

Every integer ≥ 2 is a product of primes.

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**

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2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**

Every integer ≥ 2 is a product of primes.

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   P(j) **is true for every integer** j **between** 2 **and** k

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4. **Inductive Step:**

   Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

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   <u>Case: k+1 **is prime**</u>: **Then by definition** k+1 **is a product of primes**

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   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; i.e. } k+1 \textbf{ is a product of primes}}$

   <u>Case: k+1 **is prime**</u>: **Then by definition** k+1 **is a product of primes**

   <u>Case: k+1 **is composite:**</u> **Then** k+1=ab **for some integers** a **and** b
   where $2 \leq a, b \leq k$.

Every integer ≥ 2 is a product of primes.

1. **Let** $P(n)$ **be** "n is a product of primes". **We will show that** $P(n)$ **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** $(n=2)$: 2 **is prime, so it is a product of primes.** **Therefore** $P(2)$ **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$, $P(j)$ **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

Goal: Show $P(k+1)$; **i.e.** $k+1$ **is a product of primes**

<u>Case: $k+1$ is prime</u>: **Then by definition** $k+1$ **is a product of primes**

<u>Case: $k+1$ is composite:</u> **Then** $k+1=ab$ **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH,** $P(a)$ **and** $P(b)$ **are true so we have**

$a = p_1 p_2 \cdots p_r$ **and** $b = q_1 q_2 \cdots q_s$

for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.

Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

Since $k \geq 2$, **one of these cases must happen and so** $P(k+1)$ **is true.**

Every integer ≥ 2 is a product of primes.

1. **Let** $P(n)$ **be** "n is a product of primes". **We will show that** $P(n)$ **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** $(n=2)$: 2 **is prime, so it is a product of primes. Therefore** $P(2)$ **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$, $P(j)$ **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; i.e. } k+1 \textbf{ is a product of primes}}$

   <u>**Case:** $k+1$ **is prime**</u>: **Then by definition** $k+1$ **is a product of primes**

   <u>**Case:** $k+1$ **is composite:**</u> **Then** $k+1=ab$ **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH,** $P(a)$ **and** $P(b)$ **are true so we have**

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   Thus, $k+1 = ab = p_1p_2 \cdots p_rq_1q_2 \cdots q_s$ **which is a product of primes.**

   Since $k \geq 2$, **one of these cases must happen and so** $P(k+1)$ **is true.**
5. **Thus** $P(n)$ **is true for all integers** $n \geq 2$, **by strong induction.**

Strong Induction is particularly useful when…

…we need to analyze methods that on input k make a recursive call for an input different from $k - 1$.

e.g.: Recursive Modular Exponentiation:

- For exponent $k > 0$ it made a recursive call with exponent $\mathrm{j} = k/2$ when k was even or $\mathrm{j} = k - 1$ when k was odd.

We won't analyze this particular method by strong induction, but we could.

However, we will use strong induction to analyze other functions with recursive definitions.

Recursive definitions of functions

- $F(0) = 0;\;\; F(n+1) = F(n) + 1$ **for all** $n \geq\; 0$**.**

- $G(0) = 1;\;\; G(n+1) = 2 \cdot G(n)$ **for all** $n \geq\; 0$**.**

- $0! = 1;\;\; (n+1)! = (n+1) \cdot n!$ **for all** $n \geq\; 0$**.**

- $H(0) = 1;\;\; H(n+1) = 2^{H(n)}$ **for all** $n \geq\; 0$**.**

More Recursive Definitions

Suppose that $h: \mathbb{N} \to \mathbb{R}$.

Then we have familiar summation notation:

$\sum_{i=0}^{0} h(i) = h(0)$

$\sum_{i=0}^{n+1} h(i) = h(n+1) + \sum_{i=0}^{n} h(i)$ for $n \geq 0$

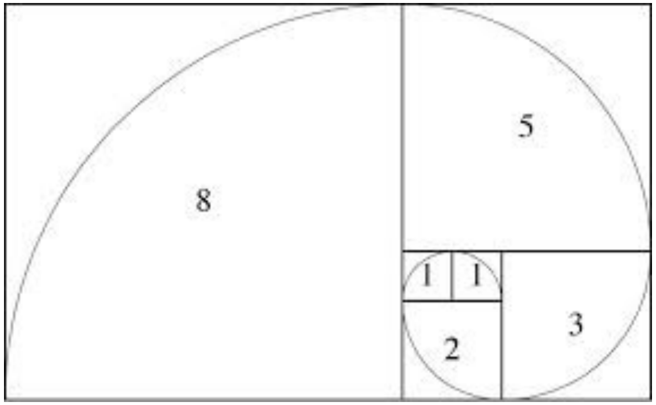
There is also product notation:

$\prod_{i=0}^{0} h(i) = h(0)$

$\prod_{i=0}^{n+1} h(i) = h(n+1) \cdot \prod_{i=0}^{n} h(i)$ for $n \geq 0$

Fibonacci Numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$



Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

$$\boldsymbol{f_0 = 0} \qquad \boldsymbol{f_1 = 1}$$

$$\boldsymbol{f_n = f_{n-1} + f_{n-2}} \text{ for all } \boldsymbol{n \geq 2}$$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**

$f_0 = 0 \qquad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0 = 0 < 1 = 2^0$ **so** P(0) **is true.**

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

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2. **Base Case:** $f_0 = 0 < 1 = 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$, **we have** $f_j < 2^j$ **for every integer** j **from** 0 **to** k.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

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4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; that is, } f_{k+1} < 2^{k+1}}$

$$\boldsymbol{f_0 = 0} \qquad \boldsymbol{f_1 = 1}$$
$$\boldsymbol{f_n = f_{n-1} + f_{n-2}} \quad \textbf{for all } \boldsymbol{n \geq 2}$$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

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2. **Base Case:** $f_0=0 < 1= 2^0$ **so** P(0) **is true.**
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   <u>Case k+1 = 1</u>:

   <u>Case k+1 ≥ 2</u>:

$$f_0 = 0 \qquad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be** "$f_n < 2^n$". **We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0 = 0 < 1 = 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$, **we have** $f_j < 2^j$ **for every integer** j **from** 0 **to** k.
4. **Inductive Step:** **Goal: Show** P(k+1); **that is,** $f_{k+1} < 2^{k+1}$

   <u>Case k+1 = 1</u>: Then $f_1 = 1 < 2 = 2^1$ **so** P(k+1) **is true here.**

   <u>Case k+1 ≥ 2</u>:

$$f_0 = 0 \qquad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0=0 < 1= 2^0$ so P(0) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, we have $f_j < 2^j$ for every integer j from 0 to k.
4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} < 2^{k+1}$

   <u>Case $k+1 = 1$</u>: Then $f_1 = 1 < 2 = 2^1$ so P(k+1) is true here.

   <u>Case $k+1 \geq 2$</u>: Then $f_{k+1} = f_k + f_{k-1}$ by definition

   $< 2^k + 2^{k-1}$ by the IH since $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) is true in this case.

   These are the only cases so P(k+1) follows.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let $P(n)$ be "$f_n < 2^n$". We prove that $P(n)$ is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0 = 0 < 1 = 2^0$ so $P(0)$ is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, we have $f_j < 2^j$ for every integer j from 0 to k.
4. Inductive Step: $\boxed{\text{Goal: Show } P(k+1)\text{; that is, } f_{k+1} < 2^{k+1}}$

   Case $k+1 = 1$: Then $f_1 = 1 < 2 = 2^1$ so $P(k+1)$ is true here.

   Case $k+1 \geq 2$: Then $f_{k+1} = f_k + f_{k-1}$ by definition

   $< 2^k + 2^{k-1}$ by the IH since $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so $P(k+1)$ is true in this case.

   These are the only cases so $P(k+1)$ follows.

5. Therefore by strong induction,

   $f_n < 2^n$ for all integers $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

$\boldsymbol{f_0 = 0} \quad \boldsymbol{f_1 = 1}$

$\boldsymbol{f_n = f_{n-1} + f_{n-2}}$ **for all** $\boldsymbol{n \geq 2}$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

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Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be** "$f_n \geq 2^{n/2 - 1}$". **We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **is true.**

$f_0 = 0 \quad f_1 = 1$
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$\boldsymbol{f_0 = 0} \quad \boldsymbol{f_1 = 1}$
$\boldsymbol{f_n = f_{n-1} + f_{n-2}}$ for all $\boldsymbol{n \geq 2}$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

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4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

$f_0 = 0 \quad f_1 = 1$

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Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

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No need for cases for the definition here:

$f_{k+1} = f_k + f_{k-1}$ **since** $k+1 \geq 2$

Now just want to apply the IH to get P(k) **and** P(k-1)

Problem: Though we can get P(k) **since** $k \geq 2$,

k-1 **may only be** 1 **so we can't conclude** P(k-1)

Solution: Separate cases for when k-1=1 **(or** k+1=3**).**

$$f_0 = 0 \qquad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2 -1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} \geq 2^{(k+1)/2 -1}$

   Case k = 2:

   Case k ≥ 3:

$f_0 = 0$ $\quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2-1}$**".** **We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
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3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 2$, P(j) **is true for every integer** j **from** 2 **to** k.
4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

   <u>Case k = 2</u>: **Then** $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$

   <u>Case k ≥ 3</u>:

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: $\boxed{\text{Goal: Show } P(k+1)\text{; that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

   Case k = 2: Then $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$

   Case k ≥ 3: $f_{k+1} = f_k + f_{k-1}$ by definition

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ by the IH since $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   So P(k+1) is true in both cases.
5. Therefore by strong induction, $f_n \geq 2^{n/2-1}$ for all integers $n \geq 0$.

$$f_0 = 0 \qquad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Running time of Euclid's algorithm

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An informal way to get the idea: Consider an n **step** gcd **calculation starting with** r_{n+1}=a **and** r_n=b:

$$
\begin{aligned}
r_{n+1} &= q_n r_n + r_{n-1} \\
r_n &= q_{n-1} r_{n-1} + r_{n-2} \\
&\ \ \dots \\
r_3 &= q_2 r_2 + r_1 \\
r_2 &= q_1 r_1
\end{aligned}
$$

For all $k \geq 2$, $r_{k-1} = r_{k+1} \bmod r_k$

Now $r_1 \geq 1$ **and each** q_k **must be** ≥ 1. **If we replace all the** q_K's **by** 1 **and replace** r_1 **by** 1 **, we can only reduce the** r_k's. **After that reduction,** $r_k = f_k$ **for every** k.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

We go by strong induction on n**.**

Let P(n) **be** "gcd(a,b) **with** $a \geq b > 0$ **takes** n **steps** $\rightarrow a \geq f_{n+1}$" **for all** $n \geq 1$**.**

Base Case: n=1 **Suppose Euclid's Algorithm with** $a \geq b > 0$ **takes** 1 **step.**
By assumption, $a \geq b \geq 1 = f_2$ **so** P(1) **holds.**

Induction Hypothesis: Suppose that for some integer $k \geq 1$, P(j) **is true for all integers** j **s.t.** $1 \leq j \leq k$

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

We go by strong induction on n.

Let P(n) be "gcd(a,b) with $a \geq b>0$ takes n steps $\rightarrow a \geq f_{n+1}$" for all $n \geq 1$.

Base Case: n=1 Suppose Euclid's Algorithm with $a \geq b > 0$ takes 1 step.
By assumption, $a \geq b \geq 1 = f_2$ so P(1) holds.

Induction Hypothesis: Suppose that for some integer $k \geq 1$, P(j) is true for all integers j s.t. $1 \leq j \leq k$

Inductive Step: We want to show: if gcd(a,b) with $a \geq b > 0$ takes k+1 steps, then $a \geq f_{k+2}$.

Running time of Euclid's algorithm

<u>Induction Hypothesis</u>: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

<u>Inductive Step</u>: **Goal: if** $\gcd(a,b)$ **with** $a \geq b > 0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2.}$

Now if $k+1=2$, **then Euclid's algorithm on** a **and** b **can be written as**

$$a = q_2 b + r_1$$
$$b = q_1 r_1$$

and $r_1 > 0$.

Also, since $a \geq b > 0$ **we must have** $q_2 \geq 1$ and $b \geq 1$.

So $a = q_2 b + r_1 \geq b + r_1 \geq 1+1 = 2 = f_3 = f_{k+2}$ as required.

Running time of Euclid's algorithm

<u>Induction Hypothesis</u>: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

<u>Inductive Step</u>: **Goal: if** $\gcd(a,b)$ **with** $a \geq b>0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2}$.

Next suppose that $k+1 \geq 3$ **so for the first 3 steps of Euclid's algorithm on** a **and** b **we have**

$$a = q_{k+1}b + r_k$$
$$b = q_k r_k + r_{k-1}$$
$$r_k = q_{k-1}r_{k-1} + r_{k-2}$$

and there are $k-2$ **more steps after this.**

Running time of Euclid's algorithm

Induction Hypothesis: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

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So since $k, k-1 \geq 1$ **by the IH we have** $b \geq f_{k+1}$ **and** $r_k \geq f_k$.

Running time of Euclid's algorithm

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So since $k, k-1 \geq 1$ **by the IH we have** $b \geq f_{k+1}$ **and** $r_k \geq f_k$.

Also, since $a \geq b$ **we must have** $q_{k+1} \geq 1$.

So $a = q_{k+1} b + r_k \geq b + r_k \geq f_{k+1} + f_k = f_{k+2}$ **as required.** ■

Running time of Euclid’s algorithm

Theorem: Suppose that Euclid’s Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Why does this help us bound the running time of Euclid’s Algorithm?

We already proved that $f_n \geq 2^{n/2 - 1}$ so $f_{n+1} \geq 2^{(n-1)/2}$

Therefore: if Euclid’s Algorithm takes n steps
for $\gcd(a, b)$ with $a \geq b > 0$
then $a \geq 2^{(n-1)/2}$

so $(n-1)/2 \leq \log_2 a$ or $n \leq 1 + 2\log_2 a$
i.e., # of steps ≤ 1 + twice the # of bits in a.