Warm up translate to predicate logic:
"For every x, if x is even, then $x = 2$."

More Quantifiers!

Lecture 6

Evaluating Predicate Logic

"For every x, if x is even, then $x = 2$." / $\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

Is this true?

Evaluating Predicate Logic

"For every x, if x is even, then $x = 2$." / $\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

Is this true?

TRICK QUESTION! It depends on the domain.

Prime Numbers	Positive Integers	Odd integers
True	False	True (vacuously)

The truth value of a quantified formula depends on the domain.

	$\{-3, 3\}$	Integers	Odd Integers
$\forall x.\, \text{Odd}(x)$	True	False	True
$\forall x.\, \text{LessThan5}(x)$	True	False	False

	$\{-3, 3\}$	Integers	Positive Multiples of 5
$\exists x.\, \text{Odd}(x)$	True	True	True
$\exists x.\, \text{LessThan5}(x)$	True	True	False

You can think of $\forall\, x.\ P(x)$ as conjunction over all objects in the domain, and $\exists x. P(x)$ as disjunction over all objects in the domain.

- $\forall x.\, \text{Odd}(x)$
  - over $\{-3, 3\}$ is the conjunction $\text{Odd}(-3) \wedge \text{Odd}(3)$
  - over integers is the infinite conjunction $\ldots \wedge \text{Odd}(-1) \wedge \text{Odd}(0) \wedge \text{Odd}(1) \wedge \ldots$
- $\exists x.\, \text{Odd}(x)$
  - over $\{-3, 3\}$ is the disjunction $\text{Odd}(-3) \vee \text{Odd}(3)$
  - over integers is the infinite disjunction $\ldots \vee \text{Odd}(-1) \vee \text{Odd}(0) \vee \text{Odd}(1) \vee \ldots$

More Practice

Let your domain of discourse be fruits sold at QFC.

There is a fruit that is tasty and ripe.

$$\exists x(\texttt{Tasty}(x) \land \texttt{Ripe}(x))$$

For every fruit, if it is not ripe then it is not tasty.

$$\forall x(\neg\texttt{Ripe}(x) \rightarrow \neg\texttt{Tasty}(x))$$

There is a fruit that is sliced and diced.

$$\exists x(\texttt{Sliced}(x) \land \texttt{Diced}(x))$$

Translation hints

Domain of discourse
Mammals

Predicate definitions
Cat(x) := “x is a cat”
Orange(x) := “x is orange”
LikesLasagna(x) := “x likes lasagna”

“Orange cats like lasagna.”

$\forall x.\ (((\text{Orange}(x))\ \text{Cat}(x)) \rightarrow\ \text{LikesLasagna}(x))$

When there's no leading quantification, it means "for all".

When restricting to a smaller domain in a "for all", use implication.

"Some orange cats don’t like lasagna."

“Some" means "there exists".

When restricting to a smaller domain in an "exists", use conjunction.

When putting predicates together, orange cats, use conjunction.

One Technical Matter

How do we parse sentences with quantifiers?

What's the "order of operations?"

We will usually put parentheses right after the quantifier and variable to make it clear what's included. If we don't, it's the rest of the expression.

Be careful with repeated variables…they don't always mean what you think they mean.

$\forall x\big(a(x)\big) \land \forall x(b(x))$ are different x's.

Bound Variables

What happens if we repeat a variable?

Whenever you introduce a new quantifier with an already existing variable, it “takes over” that name until its expression ends.

$$\forall x(a(x) \land \forall x[b(x)] \land R(x))$$

It’s common (albeit somewhat confusing) practice to reuse a variable when it “wouldn’t matter”.

Never do something like the above: where a single name switches from gold to purple back to gold. Switching from gold to purple only is usually fine…but names are cheap.

Negations of quantifiers

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Domain of discourse	**Predicate definitions**
{plum, apple, …}	PurpleFruit(x) := "x is a purple fruit"

Let P be the formula $\forall x.\ \mathrm{PurpleFruit}(x)$, "all fruits are purple."

What is the negation of P?

a. "There exists a purple fruit" ($\exists x.\ \mathrm{PurpleFruit}(x)$)
b. "There exists a non-purple fruit" ($\exists x.\ \neg\mathrm{PurpleFruit}(x)$)
c. "All fruits are not purple" ($\forall x.\ \neg\mathrm{PurpleFruit}(x)$)

DeMorgan's laws for quantifiers

$\neg \forall x.\ P(x) \equiv \exists x. \neg P(x)$

$\neg \exists x. P(x) \equiv \forall x. \neg P(x)$

Negating Quantifiers

$\neg \forall x\, P(x)$

$\exists x.\, \neg P(x)$

What happens when we negate an expression with quantifiers?

What does your intuition say?

Original	Negation
P= Every positive integer is prime	There is a positive integer that is not prime. (There is some + number which is not prime)
$\forall x$ `Prime`(x)	$\exists x(\neg$`Prime`$(x))$ $\neg \exists x. \neg$Prime(x)
Domain of discourse: positive integers	Domain of discourse: positive integers

[Satisfied for $D' = \{\} \subseteq D$] $\longrightarrow \neg$

Subsets of D only including prime

$D = \{1, 2, 3, \ldots \}$

For any subset of D, exactly one of P, ¬P, hold.

Negating Quantifiers

DoD = {positive ints}

Let's try on an existential quantifier…

Original

There is a positive integer which is prime and even.

$\exists x(\texttt{Prime}(x) \wedge \texttt{Even}(x))$
Domain of discourse: positive integers

Negation

Every positive integer is composite or odd.

x isn't prime or x isn't even.

$\forall x(\neg\texttt{Prime}(x) \vee \neg\texttt{Even}(x))$
Domain of discourse: positive integers

To negate an expression with a quantifier
1. Switch the quantifier (∀ becomes ∃, ∃ becomes ∀)
2. Negate the expression inside

Negation

Domain of Discourse = Animals

Translate these sentences to predicate logic, then negate them.

All cats have nine lives.

$\forall x.\ Cat(x) \to HasLives(x, 9)$

$$\forall x(Cat(x) \to NumLives(x,9))$$

$\exists x(Cat(x) \land \neg(NumLives(x,9)))$ "There is a cat without 9 lives."

All dogs love every person.

$$\forall x \forall y(Dog(x) \land Human(y) \to Love(x,y))$$

Holds for empty set

$\exists x \exists y(Dog(x) \land Human(y) \land \neg Love(x,y))$ "There is a dog who does not love someone." "There is a dog and a person such that the dog doesn't love that person."

There is a cat that loves someone.

$$\exists x \exists y(Cat(x) \land Human(y) \land Love(x,y))$$

$$\forall x \forall y(Cat(x) \land Human(y) \to \neg Love(x,y))$$

"For every cat and every human, the cat does not love that human."

"Every cat does not love any human" ("no cat loves any human")

Negation with Domain Restriction

$\neg(A \land B \land C)$
$\equiv \neg A \lor \neg B \lor \neg C$

De Morgan's Law

$$\exists x \exists y (Cat(x) \land Human(y) \land Love(x,y))$$
$$\forall x \forall y (Cat(x) \land Human(y) \rightarrow \neg Love(x,y))$$

"For every cat and every human, the cat does not love that human."
"Every cat does not love any human" ("no cat loves any human")

There are lots of equivalent expressions to the second. This one is by far the best because it reflects the domain restriction happening. How did we get there?

Think of statement as a conjunction over all objects in the domain, negate that conjunction, and convert back to a quantified formula.

Quantifiers, more translations

∀ (for **A**ll) and ∃ (there **E**xists)

Write these statements in predicate logic with quantifiers. Let your domain of discourse be "cats" → "mammals"

∀

This sentence implicitly makes a statement about all cats!

If a cat is fat, then it is happy.

$\forall x[\texttt{Fat}(x) \rightarrow \texttt{Happy}(x)]$

Quantifiers, implications

Writing implications can be tricky when we change the domain of discourse.

If a cat is fat, then it is happy.

Domain of Discourse: cats $\forall x[\texttt{Fat}(x) \rightarrow \texttt{Happy}(x)]$

What if we change our domain of discourse to be all mammals?

We need to limit x to be a cat. How do we do that?

$\forall x[(\texttt{Cat}(x) \wedge \texttt{Fat}(x)) \rightarrow \texttt{Happy}(x)]$ $\forall x[\texttt{Cat}(x) \wedge (\texttt{Fat}(x) \rightarrow \texttt{Happy}(x))]$

Quantifiers, implications

Which of these translates "If a cat is fat then it is happy."
when our domain of discourse is "mammals"?

$\forall x[(\texttt{Cat}(x) \land \texttt{Fat}(x)) \rightarrow \texttt{Happy}(x)]$

For all mammals, if x is a cat and fat then it is happy
[if x is not a cat, the claim is vacuously true, you can't use the promise for anything]

$\forall x[\texttt{Cat}(x) \land (\texttt{Fat}(x) \rightarrow \texttt{Happy}(x))]$

For all mammals, that mammal is a cat and if it is fat then it is happy.
[What if x is a dog? Dogs are in the domain, but...uh-oh. This isn't what we meant.]

To "limit" variables to a portion of your domain of discourse under a universal quantifier add a hypothesis to an implication.

Quantifiers, implications

Existential quantifiers need a different rule:

To "limit" variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

∃
There is a dog who is not happy.

Domain of discourse: dogs ⇝ mammals

$\exists x(\neg \texttt{Happy}(x))$

Quantifiers, existential and conjunction

Which of these translates "There is a dog who is not happy."
when our domain of discourse is "mammals"?

$\exists x[\text{Dog}(x) \rightarrow \neg\text{Happy}(x))]$

Mammals w/o dogs ↑ True

There is a mammal, such that if x is a dog then it is not happy.
[this can't be right – plug in a cat for x and the implication is true]

$\exists x[(\text{Dog}(x) \wedge \neg\text{Happy}(x)]$

~~For all mammals that mammal is a cat and if it is a cat then it is happy.
[this one is correct!]~~

dog

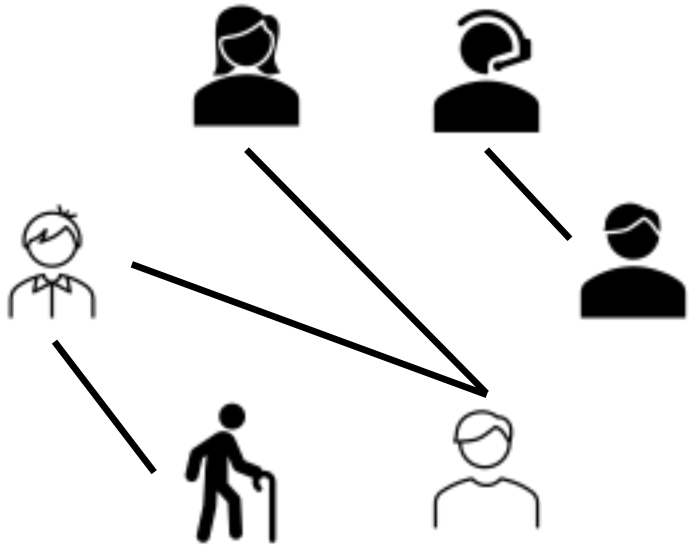
To "limit" variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

Unalike Nested Quantifiers

Translate these sentences using only quantifiers and the predicate `AreFriends`(x, y)

Everyone is friends with someone.

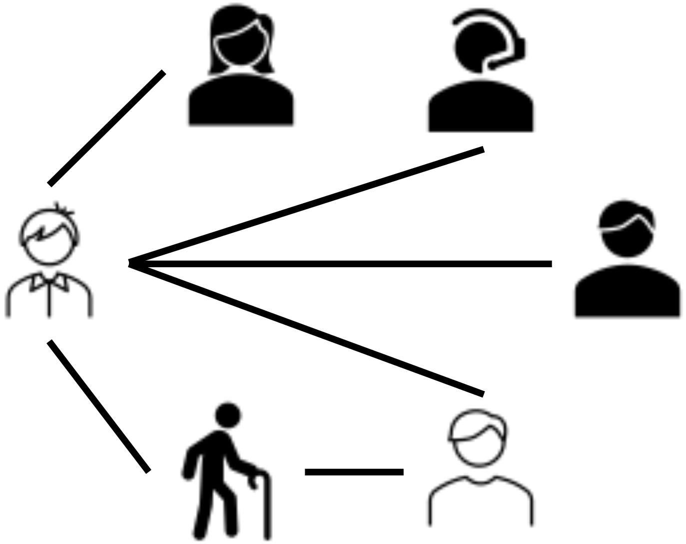
$\forall$ $\exists$



$\forall x. \exists y, \text{AreFriends}(x,y)$

Someone is friends with everyone.

$\exists$ $\forall$



$\exists x. \forall y. \text{AreFriends}(x, y)$

Unalike Nested Quantifiers

Translate these sentences using only quantifiers and the predicate $\texttt{AreFriends}(x, y)$

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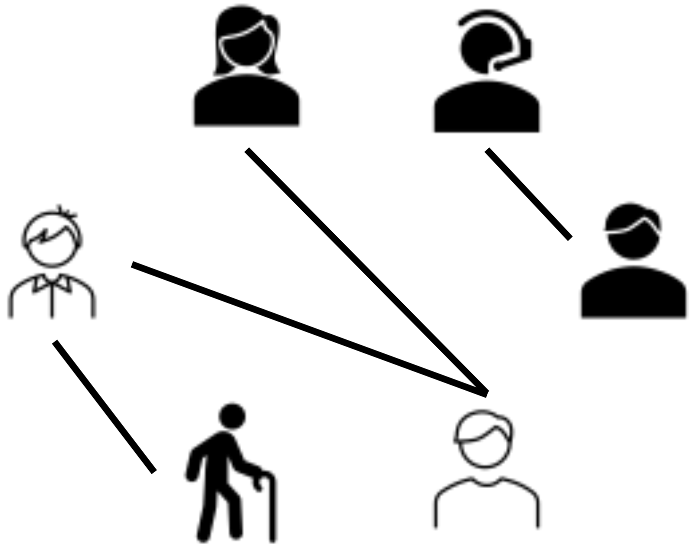

$\forall x(\exists y\ \texttt{AreFriends}(x, y))$

$\forall x \exists y\ \texttt{AreFriends}(x, y)$

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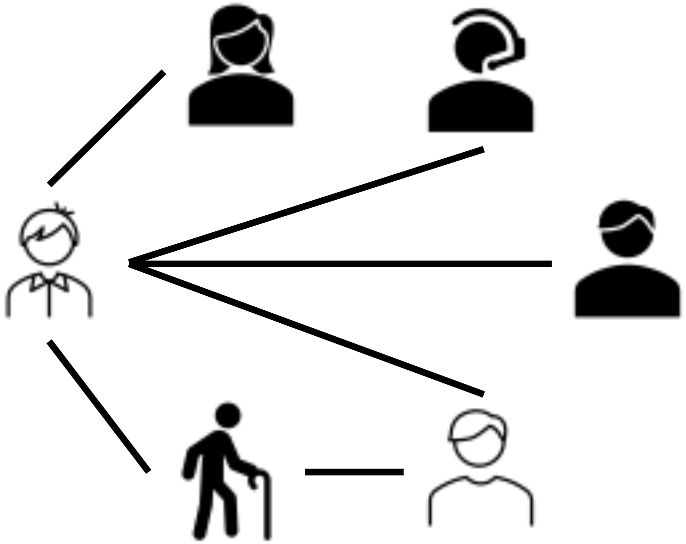


$\exists x(\forall y\ \texttt{AreFriends}(x, y))$

$\exists x \forall y\ \texttt{AreFriends}(x, y)$

Unalike Nested Quantifiers

$\forall x \exists y\ a(x, y)$

"For every x there exists a y such that $a(x, y)$ is true."

y might change depending on the x (people have different friends!).

$\exists x \forall y\ a(x, y)$

"There is an x such that for all y, $a(x, y)$ is true."

There's a special, magical x value so that $a(x, y)$ is true regardless of y.

Unalike Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $a(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y\, a(x, y)$ → Row which is all T

$\forall x \exists y\, a(x, y)$

x = A, y = A
B, y = A
C, y = B
D, y = E
E, y = D

y

$a(x, y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

x

Unalike Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $a(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y a(x, y)$

A row, where every entry is T

$\forall x \exists y a(x, y)$

In every row there must be a T

$a(x, y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Keep everything in order

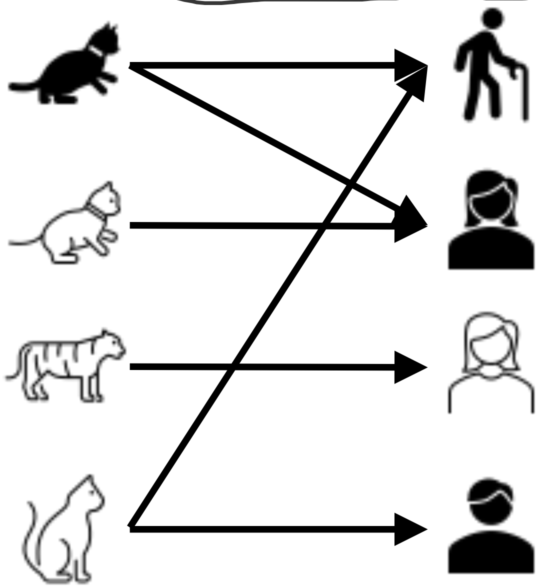
Keep the quantifiers in the same order in English as they are in the logical notation.

"There is someone out there for everyone" is a $\forall x \exists y$ statement in "everyday" English.

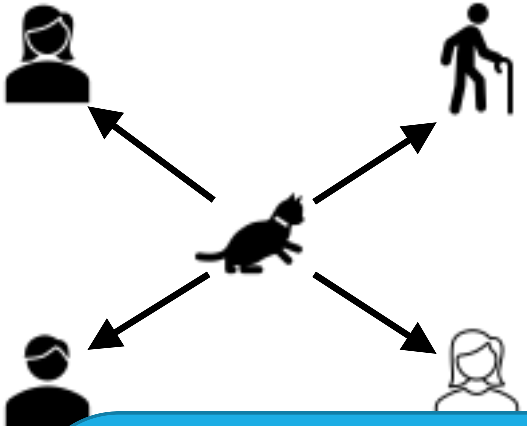
It would **never** be phrased that way in "mathematical English" We'll only every write "for every person, there is someone out there for them."

Try it yourselves

Every cat loves some human.



There is a cat that loves every human.



Let your domain of discourse be mammals.
Use the predicates `Cat`(x), `Dog`(x), and `Loves`(x, y) t

Human(z)

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Try it yourselves

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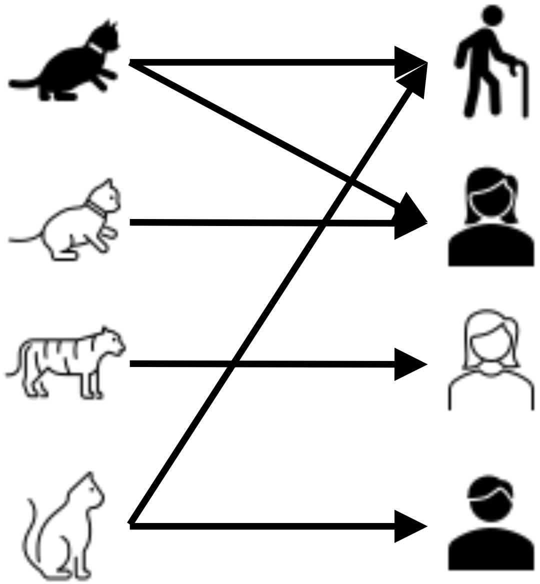
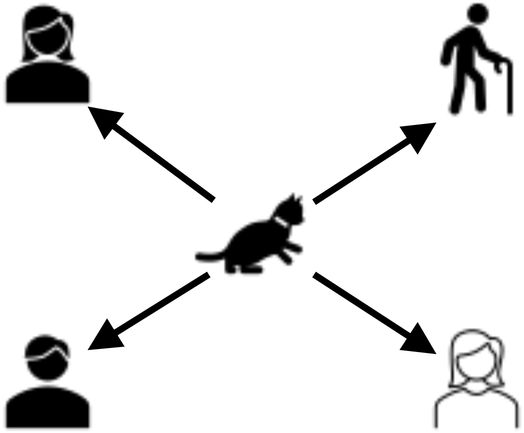


$\forall x\ (\texttt{Cat}(x) \rightarrow \exists y[\texttt{Human}(y) \wedge \texttt{Loves}(x,y)])$

$\forall x \exists y(\texttt{Cat}(x) \rightarrow [\texttt{Human}(y) \wedge \texttt{Loves}(x,y)])$

There is a cat that loves every human.



$\exists x\ (\texttt{Cat}(x) \wedge \forall y[\texttt{Human}(y) \rightarrow \texttt{Loves}(x,y)])$

$\exists x \forall y(\texttt{Cat}(x) \wedge [\texttt{Human}(y) \rightarrow \texttt{Loves}(x,y)])$

Negation

How do we negate nested quantifiers?

The old rule still applies.

DeMorgan: to negate an expression with a quantifier
1. Switch the quantifier (∀ becomes ∃, ∃ becomes ∀)
2. Negate the expression inside

$\neg(\forall x \exists y \forall z\, [a(x,y) \wedge b(y,z)])$

$\exists x(\neg(\exists y \forall z\, [a(x,y) \wedge b(y,z)]))$

$\exists x \forall y(\neg(\forall z[a(x,y) \wedge b(y,z)]))$

$\exists x \forall y \exists z(\neg[a(x,y) \wedge b(y,z)])$

$\exists x \forall y \exists z[\neg a(x,y) \vee \neg b(y,z)]$

DeMorgan's laws for quantifiers

$\neg \forall x.\ P(x) \equiv \exists x. \neg P(x)$

$\neg \exists x. P(x) \equiv \forall x. \neg P(x)$

"There is no largest integer."

$\neg \exists x. \forall y.\ (x \geq y)$

$\equiv \forall x. \neg \forall y.\ (x \geq\ y)$ DeMorgan

$\equiv \forall x. \exists y.\ \neg (x \geq\ y)$ DeMorgan

$\equiv \forall x. \exists y.\ (x\ <\ y)$ Semantics of $>$

"For every integer there is a larger integer."

More Translation

For each of the following, translate it, then say whether the statement is true. Let your domain of discourse be integers.

For every integer, there is a greater integer.

$\forall x \exists y (\texttt{Greater}(y, x))$ (This statement is true: y can be $x + 1$ [y depends on x])

There is an integer x, such that for all integers y, xy is equal to 1.

$\exists x \forall y (\texttt{Equal}(xy, 1))$ (This statement is false: no single value of x can play that role for every y.)

$\forall y \exists x (\texttt{Equal}(x + y, 1))$

For every integer, y, there is an integer x such that $x + y = 1$
(This statement is true, y can depend on x