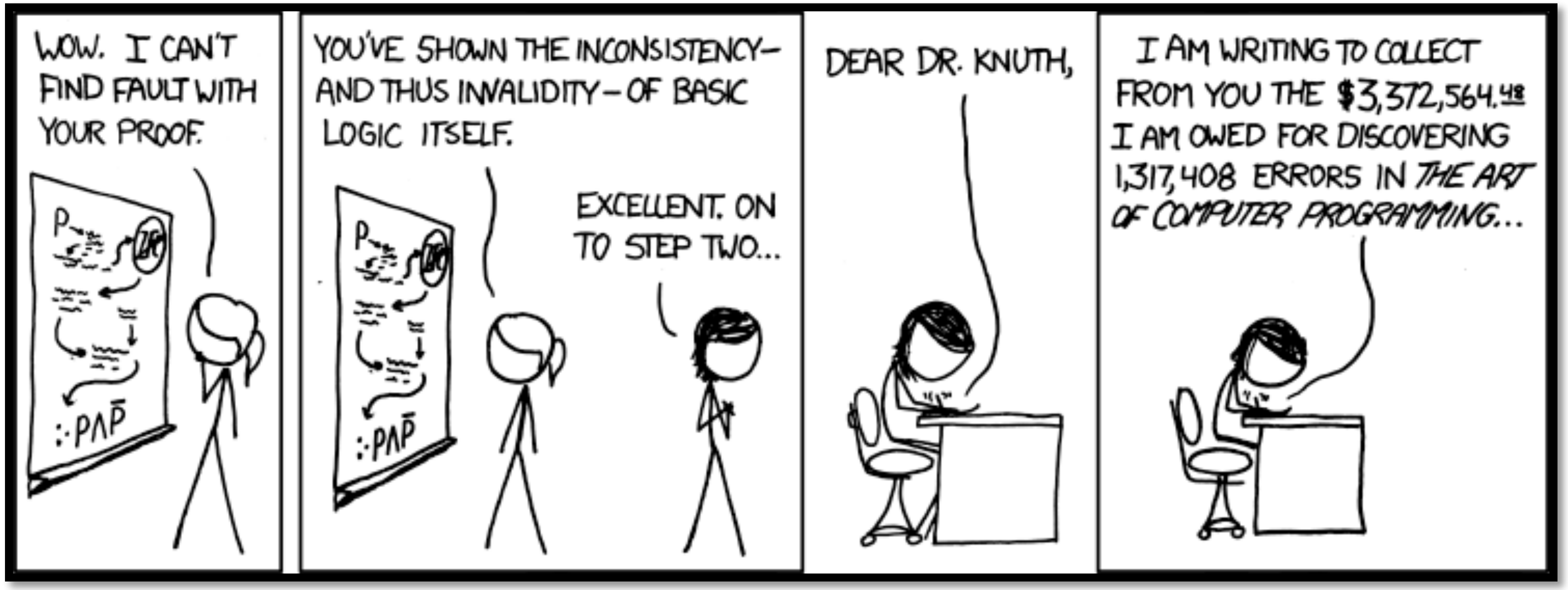

xkcd.com/816/

If early, download today's activity slide from the website☺

Inference Proofs, With Quantifiers

Announcements

Proof checking tool: https://homes.cs.washington.edu/~kevinz/proof-test/

Will check your symbolic proofs, so you know if you've applied rules properly. – I do recommend it for rough drafts, I don't recommend for when you're "stuck"

About Grades

Grades were critical in your lives up until now.

- If you were in high school, they're critical for getting into college.
- If you were at UW applying to CSE, they were key to that application

Regardless of where you're going next, what you **learn** in this course matters FAR more than what your grade in this course.

If you're planning on industry – interviews matter more than grades.

If you're planning on grad school – letters matter most, those are based on doing work outside of class building off what you learned in class.

About Grades

What that means:

The TAs and I are going to prioritize your learning over debating whether -2 or -1 is “ more fair”

If you’re worried about “have I explained enough” – write more!

It’ll take you longer to write the Ed question than write the extended answer. We don’t take off for too much work.

And the extra writing is going to help you learn more anyway.

Regrades

TAs make mistakes!

When I was a TA, I made errors on 1 or 2% of my grading that needed to be corrected. If we made a mistake, file a regrade request on gradescope.

But those are only for mistakes, not for whether "-1 would be more fair"

If you are confused, please talk to us!

My favorite office hours questions are "can we talk about the best way to do something on the homework we just got back?"

If **after** you do a regrade request on gradescope, you still think a grading was incorrect, send email to staff.

Regrade requests will close 2 weeks after homework is returned.

In general

How do you convince someone that $a \rightarrow b$ is true given some surrounding context/some surrounding givens?

You suppose a is true (you assume a)

And then you'll show b must also be true.

Just from a and the Given information.

We need an additional connective

Suppose we've written a proof that A implies B (using either inference rules or a proof in predicate logic),
possibly using some assumptions

$\underline{A \to B}$
Proposition

How do we annotate the relationship between A and B?

$$\boxed{A \Rightarrow B}$$

The Direct Proof Rule

We've been implicitly using another "rule", the direct proof rule

$$\frac{\text{Write a proof "given } A \text{ conclude } B\text{"}}{A \to B}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \to B}$$

This rule is different from the others – $A \Rightarrow B$ is not a "single fact."
It's an observation that we've done a proof. (i.e. that we showed fact B starting from A.)

We will get a lot of mileage out of this rule…

Inference Rules

Eliminate ∧

$$\frac{A \wedge B}{\therefore \quad A, B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∧

$$\frac{A; B}{\therefore \quad A \wedge B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens

$$\frac{a \rightarrow b; a}{\therefore \quad b}$$

You can still use all the propositional logic equivalences too!

Caution

Be careful! Logical inference rules can only be applied to **entire** facts. They cannot be applied to portions of a statement (the way our propositional rules could). Why not?

Suppose we know $a \to b$, r. Can we conclude b?

1.	$a \to b$	Given
2.	r	Given
3.	$(a \vee r) \to b$	Introduce ∨ (1)
4.	$a \vee r$	Introduce ∨ (2)
5.	b	Modus Ponens 3,4.

$$\text{Intro } \vee \quad \frac{A}{\therefore A \vee B, B \vee A}$$

One more Proof

Show if we know: $a, b, [(a \wedge b) \rightarrow (r \wedge s)], r \rightarrow t$ we can conclude t.

One more Proof

Show if we know: $a, b, [(a \wedge b) \rightarrow (r \wedge s)], r \rightarrow t$ we can conclude t.

1.	a	Given
2.	b	Given
3.	$[(a \wedge b) \rightarrow (r \wedge s)]$	Given
4.	$r \rightarrow t$	Given
5.	$a \wedge b$	Intro $\wedge$ (1,2)
6.	$r \wedge s$	Modus Ponens (3,5)
7.	r	Eliminate $\wedge$ (6)
8.	t	Modus Ponens (4,7)

introduce our assumptions

Given: $((a \to b) \land (b \to r))$
Show: $(a \to r)$

Here's an incorrect proof.

1.	$(a \to b) \land (b \to r)$	Given
2.	$a \to b$	Eliminate ∧ (1)
3.	$b \to r$	Eliminate ∧ (1)
4.	a	Given???
5.	b	Modus Ponens 4,2
6.	r	Modus Ponens 5,3
7.	$a \to r$	Direct Proof Rule

[Assume a, deduce r]

Given: $((a \to b) \land (b \to r))$
Show: $(a \to r)$

Here's an incorrect proof.

Proofs are supposed to be lists of facts.
Some of these "facts" aren't really facts...

1. $(a \to b) \land (b \to r)$
2. $a \to b$
3. $b \to r$ Eliminate $\land$ (1)
4. a Given ????
5. b Modus Ponens 4,2
6. r Modus Ponens 5,3
7. $a \to r$ Direct Proof Rule

These facts depend on a.
But a isn't known generally.
It was assumed for the purpose of proving $a \to r$.

Given: $((a \to b) \land (b \to r))$
Show: $(a \to r)$

Here's an incorrect proof.

Proofs are supposed to be lists of facts.
Some of these "facts" aren't really facts…

1. $(a \to b) \land (b \to r)$
2. $a \to b$
3. $b \to r$ — Eliminate ∧ (1)
4. a — Given ????
5. b — Modus Ponens 4,2
6. r — Modus Ponens 5,3
7. $a \to r$ — Direct Proof Rule

These facts depend on a.
But a isn't known generally.
It was assumed for the purpose of proving $a \to r$.

Given: $((a \to b) \land (b \to r))$ → consider when this is true

Show: $(a \to r)$ & $a \to r$ is also true

Here's a corrected version of the proof.

1. $(a \to b) \land (b \to r)$	Given	
2. $a \to b$	Eliminate ∧ 1	
3. $b \to r$	Eliminate ∧ 1	
4.1 a	Assumption (truth value of a is true)	
4.2 b (a)	Modus Ponens 4.1,2	
4.3 r (b(a))	Modus Ponens 4.2,3	
5. $a \to r$	Direct Proof Rule	

When introducing an assumption to prove an implication: Indent, and change numbering.

When reached your conclusion, use the Direct Proof Rule to observe the implication is a fact.

The conclusion is an unconditional fact (doesn't depend on a) so it goes back up a level

Try it!

Given: $a \vee b$, $(r \wedge s) \rightarrow \neg b$, r.
Show: $s \rightarrow a$

Eliminate ∧	$A \wedge B$	$\therefore\ A, B$
Eliminate ∨	$A \vee B, \neg A$	$\therefore\ B$
Intro ∧	$A; B$	$\therefore\ A \wedge B$
Intro ∨	A	$\therefore A \vee B, B \vee A$

$\times$ No $\hat{\wedge}$ $\wedge$ to eliminate

Direct Proof rule	$A \Rightarrow B$	$A \rightarrow B$
Modus Ponens	$P \rightarrow Q; P$	$\therefore\ Q$

You can still use all the propositional logic equivalences too!

Try it!

Given: $a \vee b$, $(r \wedge s) \rightarrow \neg b$, r.
Show: $s \rightarrow a$

1.	$a \vee b$	Given
2.	$(r \wedge s) \rightarrow \neg b$	Given
3.	r	Given
4.1	s	Assumption
4.2	$r \wedge s$	Intro $\wedge$ (3,4.1)
4.3	$\neg b$	Modus Ponens (2, 4.2)
4.4	$b \vee a$	Commutativity (1)
4.5	a	Eliminate $\vee$ (4.4, 4.3)
5.	$s \rightarrow a$	Direct Proof Rule

Inference Rules

Eliminate ∧

$$\frac{A \wedge B}{\therefore\ A, B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore\ B}$$

Intro ∧

$$\frac{A; B}{\therefore\ A \wedge B}$$

Intro ∨

$$\frac{A}{\therefore\ A \vee B, B \vee A}$$

Excluded Middle

$$\frac{}{\therefore\ A \vee \neg A}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens

$$\frac{P \rightarrow Q; P}{\therefore\ Q}$$

You can still use all the propositional logic equivalences too!

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore\ P(c) \text{ for a fresh } c}$$

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore\ P(a) \text{ for any } a}$$

Intro ∀

$$\frac{P(a); a \text{ is arbitrary}}{\therefore\ \forall x\, P(x)}$$

DeMorgan's (Quantifiers)

$$\neg(\forall x\, A) \equiv \exists x(\neg A)$$
$$\neg(\exists x A) \equiv \forall x(\neg A)$$

Try it!

Given: $a \vee b$, $(r \wedge s) \rightarrow \neg b$, r.
Show: $s \rightarrow a$

1.	$a \vee b$	Given
2.	$(r \wedge s) \rightarrow \neg b$	Given
3.	r	Given
4.1	s	Assumption
4.2	$r \wedge s$	Intro $\wedge$ (3,4.1)
4.3	$\neg b$	Modus Ponens (2, 4.2)
4.4	$b \vee a$	Commutativity (1)
4.5	a	Eliminate $\vee$ (4.4, 4.3)
5.	$s \rightarrow a$	Direct Proof Rule

$S \rightarrow r \wedge S$

$S \rightarrow \neg b$

$S \rightarrow b \vee a$

$S \rightarrow a$

Try it!

Given: $a \vee b$, $(r \wedge s) \rightarrow \neg b$, r.
Show: $s \rightarrow a$

1.	$a \vee b$	Given
2.	$(r \wedge s) \rightarrow \neg b$	Given
3.	r	Given
4.1	s	Assumption
4.2	$r \wedge s$	Intro $\wedge$ (3,4.1)
4.3	$\neg b$	Modus Ponens (2, 4.2)
4.4	$b \vee a$	Commutativity (1)
4.5	a	Eliminate $\vee$ (4.4, 4.3)
5.	$s \rightarrow a$	Direct Proof Rule

Proofs with Quantifiers

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀

$$\frac{\forall x\; a(x)}{\therefore\; a(a) \text{ for any } a}$$

Intro ∃

$$\frac{a(c) \text{ for some } c}{\therefore\; \exists x\; a(x)}$$

Intro ∀

$$\frac{a(\cancel{x});\; \cancel{x} \text{ is } \textbf{arbitrary}}{\therefore\; \forall x\; a(x)}$$

Eliminate ∃

$$\frac{\exists x a(x)}{\therefore\; a(c) \text{ for a } \textbf{fresh } c}$$

Let's see a good example, then come back to those "arbitrary" and "fresh" conditions.

Proof Using Quantifiers

Suppose we know $\exists x a(x)$ and $\forall y[\, a(y) \rightarrow b(y)]$. Conclude $\exists x b(x)$.

Eliminate ∀

$$\frac{\forall x\; a(x)}{\therefore\; a(a) \text{ for any } a}$$

Intro ∃

$$\frac{a(c) \text{ for some } c}{\therefore\quad \exists x\; a(x)}$$

Intro ∀

$$\frac{a(a);\, a \text{ is } \textbf{arbitrary}}{\therefore\qquad \forall x\; a(x)}$$

Eliminate ∃

$$\frac{\exists x a(x)}{\therefore\; a(c) \text{ for a } \textbf{fresh } c}$$

Proof Using Quantifiers

Suppose we know $\exists x a(x)$ and $\forall y[\, a(y) \rightarrow b(y)]$. Conclude $\exists x b(x)$.

Intro ∃

$$\frac{a(c) \text{ for some } c}{\therefore \quad \exists x\, a(x)}$$

Eliminate ∃

$$\frac{\exists x a(x)}{\therefore \quad a(c) \text{ for a } \textbf{fresh } c}$$

Eliminate ∀

$$\frac{\forall x\, a(x)}{\therefore\ a(a) \text{ for any } a}$$

Intro ∀

$$\frac{a(a);\, a \text{ is } \textbf{arbitrary}}{\therefore \quad \forall x\, a(x)}$$

Proof Using Quantifiers

Suppose we know $\exists x a(x)$ and $\forall y[\, a(y) \to b(y)]$. Conclude $\exists x b(x)$.

1.	$\exists x a(x)$	Given
2.	$a(a)$	Eliminate ∃ 1
3.	$\forall y[a(y) \to b(y)]$	Given
4.	$a(a) \to b(a)$	Eliminate ∀ 3
5.	$b(a)$	Modus Ponens 2,4
6.	$\exists x b(x)$	Intro ∃ 5

Intro ∃

$$\frac{a(c) \text{ for some } c}{\therefore \quad \exists x\, a(x)}$$

Eliminate ∃

$$\frac{\exists x a(x)}{\therefore \quad a(c) \text{ for a } \textbf{fresh } c}$$

Eliminate ∀

$$\frac{\forall x\, a(x)}{\therefore\, a(a) \text{ for any } a}$$

Intro ∀

$$\frac{a(a); a \text{ is } \textbf{arbitrary}}{\therefore \quad \forall x\, a(x)}$$

Proofs with Quantifiers

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀

$$\frac{\forall x\ a(x)}{\therefore\ a(a) \text{ for any } a}$$

Intro ∃

$$\frac{a(c) \text{ for some } c}{\therefore\quad \exists x\ a(x)}$$

Intro ∀

$$\frac{a(a);\ a \text{ is } \textbf{arbitrary}}{\therefore\qquad \forall x\ a(x)}$$

Eliminate ∃

$$\frac{\exists x a(x)}{\therefore\ \ a(c) \text{ for a } \textbf{fresh } c}$$

"arbitrary" means a is "just" a variable in our domain.
It doesn't depend on any other variables and wasn't introduced with other information.

Proofs with Quantifiers

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀

$$\frac{\forall x\, a(x)}{\therefore\ a(a) \text{ for any } a}$$

Intro ∃

$$\frac{a(c) \text{ for some } c}{\therefore\ \exists x\, a(x)}$$

Intro ∀

$$\frac{a(a);\, a \text{ is } \textbf{arbitrary}}{\therefore\ \forall x\, a(x)}$$

Eliminate ∃

$$\frac{\exists x a(x)}{\therefore\ a(c) \text{ for a } \textbf{fresh } c}$$

"fresh" means c is a new symbol (there isn't another c somewhere else in our proof).

Fresh and Arbitrary

Suppose we know $\exists x a(x)$. Can we conclude $\forall x a(x)$?

1. $\exists x\, a(x)$ Given
2. $a(a)$ Eliminate $\exists$ (1)
3. $\forall x\, a(x)$ Intro $\forall$ (2)

This proof is **definitely** wrong.
(take $a(x)$ to be "is a prime number")

a wasn't **arbitrary**. We knew something about it – it's the x that exists to make $a(x)$ true.

Intro $\exists$

$$\frac{a(c) \text{ for some } c}{\therefore\ \exists x\, a(x)}$$

Eliminate $\exists$

$$\frac{\exists x a(x)}{\therefore\ a(c) \text{ for a } \textbf{fresh } c}$$

Eliminate $\forall$

$$\frac{\forall x\, a(x)}{\therefore\ a(a) \text{ for any } a}$$

Intro $\forall$

$$\frac{a(a);\, a \text{ is } \textbf{arbitrary}}{\therefore\ \forall x\, a(x)}$$

Fresh and Arbitrary

Intro ∀

$$\frac{a(a);\ a \text{ is } \textbf{arbitrary}}{\therefore \quad \forall x\ a(x)}$$

Eliminate ∃

$$\frac{\exists x a(x)}{\therefore \quad a(c) \text{ for a } \textbf{fresh } c}$$

You can trust a variable to be **arbitrary** if you introduce it as such.

If you eliminated a ∀ to create a variable, that variable is arbitrary. Otherwise it's not arbitrary – it depends on something.

You can trust a variable to be **fresh** if the variable doesn't appear anywhere else (i.e. just use a new letter)

Fresh and Arbitrary

Eliminate ∀
$$\frac{\forall x\, a(x)}{\therefore a(a) \text{ for any } a}$$

Intro ∃
$$\frac{a(c) \text{ for some } c}{\therefore \quad \exists x\, a(x)}$$

There are no similar concerns with these two rules.

Want to reuse a variable when you eliminate ∀? Go ahead.

Have a c that depends on many other variables, and want to intro ∃?

Also not a problem.

Arbitrary

In section yesterday, you said: $[\exists y \forall x\ a(x, y)] \rightarrow [\forall x \exists y\ a(x, y)]$. Let's prove it!!

Arbitrary

In section yesterday, you said: $[\exists y \forall x\ a(x,y)] \rightarrow [\forall x \exists y\ a(x,y)]$. Let's prove it!!

1.1 $\exists y \forall x\ a(x,y)$	Assumption
1.2 $\forall x\ a(x,c)$	Elim $\exists$ (1.1)
1.3 Let a be arbitrary.	--
1.4 $a(a,c)$	Elim $\forall$ (1.2)
1.5 $\exists y\ a(a,y)$	Intro $\exists$ (1.4)
1.6 $\forall x \exists y\ a(x,y)$	Intro $\forall$ (1.5)

2. $[\exists y \forall x\ a(x,y)] \rightarrow [\forall x \exists y\ a(x,y)]$ Direct Proof Rule

Arbitrary

In section yesterday, you said: $[\exists y \forall x\ a(x,y)] \rightarrow [\forall x \exists y\ a(x,y)]$. Let's prove it!!

1.1 $\exists y \forall x\ a(x,y)$ Assumption
1.2 $\forall x\ a(x,c)$ Elim $\exists$ (1.1)

1.4 $a(a,c)$ Elim $\forall$ (1.2)
1.5 $\exists y\ a(a,y)$ Intro $\exists$ (1.4)
1.6 $\forall x \exists y\ a(x,y)$ Intro $\forall$ (1.5)
2. $[\exists y \forall x\ a(x,y)] \rightarrow [\forall x \exists y\ a(x,y)]$ Direct Proof Rule

Find The Bug

Let your domain of discourse be integers.
We claim that given $\forall x \exists y\ \texttt{Greater}(y,x)$, we can conclude $\exists y \forall x\ \texttt{Greater}(y,x)$
Where $\texttt{Greater}(y,x)$ means $y > x$

1.	$\forall x \exists y\ \texttt{Greater}(y,x)$	Given
2.	Let a be an arbitrary integer	--
3.	$\exists y\ \texttt{Greater}(y,a)$	Elim $\forall$ (1)
4.	$b \geq a$	Elim $\exists$ (2)
5.	$\forall x\ \texttt{Greater}(b,x)$	Intro $\forall$ (4)
6.	$\exists y \forall x\ \texttt{Greater}(y,x)$	Intro $\exists$ (5)

Find The Bug

1.	$\forall x \exists y$ `Greater`(y, x)	Given
2.	Let a be an arbitrary integer	--
3.	$\exists y$ `Greater`(y, a)	Elim ∀ (1)
4.	$b \geq a$	Elim ∃ (2)
5.	$\forall x$ `Greater`(b, x)	Intro ∀ (4)
6.	$\exists y \forall x$ `Greater`(y, x)	Intro ∃ (5)

b is not arbitrary. The variable b depends on a. Even though a is arbitrary, b is not!

Bug Found

There's one other "hidden" requirement to introduce $\forall$.

"No other variable in the statement can depend on the variable to be generalized"

Think of it like this -- b was probably $a + 1$ in that example.

You wouldn't have generalized from $\texttt{Greater}(a + 1, a)$

To $\forall x\ \texttt{Greater}(a + 1, x)$. There's still an a, you'd have replaced all the a's.

x depends on y if y is in a statement when x is introduced.

This issue is much clearer in English proofs, which we'll start next time.