



Announcements:

HW5 has a couple questions which have been converted to extra credit

Exam... see next slide.

Induction

CSE 311 Autumn 20
Lecture 14

Exam: We will randomly assign you to a group of size 4;

- You are welcome to collaborate within that group
- You may not collaborate with other people or groups.
- Writing up your own solution according to our whiteboard policy still required.
- You can refer to this class's materials but you may not refer to other textbooks/other courses' materials.
- Ed will be set so that students can only ask private posts during the exam; we will intermittently make announcements for clarifications via Ed. We will answer clarifying questions, but content-related questions will not be answered.
- **Any evidence that you collaborated between groups, posted a question related to our exam on any forum or discussion board (besides clarifying private questions on Ed), or referred to external materials, will result in a 0 on the entire exam.**
- Dates of exam: Tuesday 2/16 at 12:00 am through Thursday 2/18 at 11:59 pm of next week.
 - No class on Wednesday.
 - We will announce your groups on Friday.



How do we know recursion works?

```
//Assume i is a nonnegative integer
//returns 2^i.
public int CalculatesTwoToTheI(int i) {
    if(i == 0)
        return 1;
    else
        return 2*CaclulatesTwoToTheI(i-1);
}
```

Why does `CalculatesTwoToTheI(4)` calculate 2^4 ?

Convince the other people in your room

Induction

Your new favorite proof technique!

How do we show $\forall n, P(n)$?

Show $P(0)$

Show $\forall k(P(k) \rightarrow P(k + 1))$

Induction

```
//Assume i is a nonnegative integer
public int CalculatesTwoToTheI(int i){
    if(i == 0)
        return 1;
    else
        return 2*CaclulatesTwoToTheI(i-1);
}
```

Let $P(i)$ be "CalculatesTwoToTheI (i) " returns 2^i .

Note that if the input i is 0, then the if-statement evaluates to true, and $1 = 2^0$ is returned, so $P(0)$ is true.

Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Making Induction Proofs Pretty

Let $P(i)$ be “`CalculatesTwoToTheI(i)`” returns 2^i .

Base Case ($i = 0$) Note that if the input i is 0, then the if-statement evaluates to true, and $1 = 2^0$ is returned, so $P(0)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

Inductive Step: Since $k \geq 0, k \geq 1$, so the code goes to the recursive case. We will return $2 \cdot \text{CalculatesTwoToTheI}(k)$. By Inductive Hypothesis,

$\text{CalculatesTwoToTheI}(k) = 2^k$. Thus we return $2 \cdot 2^k = 2^{k+1}$.

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n .
2. Show $P(0)$ i.e. show the base case
3. Suppose $P(k)$ for an arbitrary k .
4. Show $P(k + 1)$ (i.e. get $P(k) \rightarrow P(k + 1)$)
5. Conclude by saying $P(n)$ is true for all n by induction.

Some Other Notes

Always state where you use the inductive hypothesis when you're using it in the inductive step.

It's usually the key step, and the reader really needs to focus on it.

Be careful about what values you're assuming the Inductive Hypothesis for – the smallest possible value of k should assume the base case but nothing more.

The Principle of Induction (formally)

Principle of
Induction

$P(0); \forall k(P(k) \rightarrow P(k + 1))$

\therefore

$\forall n(P(n))$

Informally: if you knock over one domino, and every domino knocks over the next one, then all your dominoes fell over.

More Induction

Induction doesn't **only** work for code!

Show that $\sum_{i=0}^n 2^i = 1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$.

More Induction

Induction doesn't **only** work for code!

Show that $\sum_{i=0}^n 2^i = 1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$.

Let $P(n) = \text{"}\sum_{i=0}^n 2^i = 2^{n+1} - 1\text{"}$

We show $P(n)$ holds for all n by induction on n .

Base Case ()

Inductive Hypothesis:

Inductive Step:

$P(n)$ holds for all $n \geq 0$ by the principle of induction.

More Induction

Induction doesn't **only** work for code!

Show that $\sum_{i=0}^n 2^i = 1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$.

Let $P(n) = \text{"}\sum_{i=0}^n 2^i = 2^{n+1} - 1\text{"}$

We show $P(n)$ holds for all n by induction on n .

Base Case ($n = 0$) $\sum_{i=0}^0 2^i = 1 = 2 - 1 = 2^{0+1} - 1$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

Inductive Step: We show $P(k + 1)$. Consider the summation $\sum_{i=0}^{k+1} 2^i = 2^{k+1} + \sum_{i=0}^k 2^i = 2^{k+1} + 2^{k+1} - 1$, where the last step is by IH.

Simplifying, we get: $\sum_{i=0}^{k+1} 2^i = 2^{k+1} + 2^{k+1} - 1 = 2 \cdot 2^{k+1} - 1 = 2^{(k+1)+1} - 1$.

$P(n)$ holds for all $n \geq 0$ by the principle of induction.

Let's Try Another Induction Proof

$$\text{Let } g(n) = \begin{cases} 2 & \text{if } n = 2 \\ g(n-1)^2 + 3g(n-1) & \text{if } n > 2 \end{cases}$$

Prove $g(n)$ is even for all $n \geq 2$ by induction on n .

Let's just set this one up, we'll leave the individual pieces as exercises.

Setup

Let $P(n)$ be " $g(n)$ is even."

HEY WAIT -- $P(0)$ isn't true $g(0)$ isn't even defined!

We can move the "starting line"

Change the base case, and then update the IH to have the smallest value of k assume just the base case.

Setup

Let $P(n)$ be " $g(n)$ is even."

We show $P(n)$ for all $n \geq 2$ by induction on n .

Base Case ($n = 2$): $g(n) = 2$ by definition. 2 is even, so we have $P(2)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: We show $P(k + 1)$. Consider $g(k + 1)$. By definition of $g(\cdot)$, $g(k + 1) = g(k)^2 + 3g(k)$. By inductive hypothesis, $g(k)$ is even, so it equals $2j$ for some integer j . Plugging in we have:

$$g(k + 1) = (2j)^2 + 3(2j) = 2(2j^2) + 2(3j) = 2(2j^2 + 3j).$$

Since j is an integer, $2j^2 + 3j$ is also an integer, and $g(k + 1)$ is even.

Therefore, $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n .
2. Base Case: Show $P(b)$ i.e. show the base case
3. Inductive Hypothesis: Suppose $P(k)$ for an arbitrary $k \geq b$.
4. Inductive Step: Show $P(k + 1)$ (i.e. get $P(k) \rightarrow P(k + 1)$)
5. Conclude by saying $P(n)$ is true for all $n \geq b$ by the principle of induction.

Let's Try Another Induction Proof

Fundamental Theorem of Arithmetic

Every positive integer greater than 1 has a unique prime factorization.

Uniqueness is hard. Let's just show existence.

I.e.

Claim: Every positive integer greater than 1 can be written as a product of primes.

Induction on Primes.

Let $P(i)$ be " i can be written as a product of primes."

We show $P(n)$ for all $n \geq 2$ by induction on n .

Base Case ($n = 2$): 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary integer $k \geq 2$.

Inductive Step:

Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

Case 2, $k + 1$ is composite:

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

We're Stuck

We can divide $k + 1$ up into smaller pieces (say s, t such that $st = k + 1$ with $2 \leq s < k + 1$ and $2 \leq t < k + 1$)

Is $P(s)$ true? Is $P(t)$ true?

I mean...it would be...

But in the inductive step we don't have it...

Let's add it to our inductive hypothesis.

Induction on Primes

Let $P(i)$ be " i can be written as a product of primes."

We show $P(n)$ for all $n \geq 2$ by induction on n .

Base Case ($n = 2$): 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

Inductive Hypothesis:

Inductive Step:

Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

Case 2, $k + 1$ is composite:

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

Induction on Primes

Let $P(i)$ be " i can be written as a product of primes."

We show $P(n)$ for all $n \geq 2$ by induction on n .

Base Case ($n = 2$): 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

Inductive Hypothesis: Suppose $P(2), \dots, P(k)$ hold for an arbitrary integer $k \geq 2$.

Inductive Step:

Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

Case 2, $k + 1$ is composite: We can write $k + 1 = st$ for s, t nontrivial divisors (i.e. $2 \leq s < k + 1$ and $2 \leq t < k + 1$). By inductive hypothesis, we can write s as a product of primes $p_1 \cdots p_j$ and t as a product of primes $q_1 \cdots q_\ell$. Multiplying these representations, $k + 1 = p_1 \cdots p_j \cdot q_1 \cdots q_\ell$, which is a product of primes.

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

Strong Induction

That hypothesis where we assume $P(\text{base case}), \dots, P(k)$ instead of just $P(k)$ is called a strong inductive hypothesis.

Strong induction is the same fundamental idea as weak ("regular") induction.

$P(0)$ is true.

And $P(0) \rightarrow P(1)$, so $P(1)$.

And $P(1) \rightarrow P(2)$, so $P(2)$.

And $P(2) \rightarrow P(3)$, so $P(3)$.

And $P(3) \rightarrow P(4)$, so $P(4)$.

...

$P(0)$ is true.

And $P(0) \rightarrow P(1)$, so $P(1)$.

And $[P(0) \wedge P(1)] \rightarrow P(2)$, so $P(2)$.

And $[P(0) \wedge \dots \wedge P(2)] \rightarrow P(3)$, so $P(3)$.

And $[P(0) \wedge \dots \wedge P(3)] \rightarrow P(4)$, so $P(4)$.

...

Making Induction Proofs Pretty

All of our **strong** induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n .
2. Base Case: Show $P(b)$ i.e. show the base case
3. Inductive Hypothesis: Suppose $P(b) \wedge \dots \wedge P(k)$ for an arbitrary $k \geq b$.
4. Inductive Step: Show $P(k + 1)$ (i.e. get $[P(b) \wedge \dots \wedge P(k)] \rightarrow P(k + 1)$)
5. Conclude by saying $P(n)$ is true for all $n \geq b$ by the principle of induction.

Strong Induction vs. Weak Induction

Think of strong induction as “my recursive call might be on LOTS of smaller values” (like mergesort – you cut your array in half)

Think of weak induction as “my recursive call is always on one step smaller.”

Practical advice:

A strong hypothesis isn't wrong when you only need a weak one (but a weak one is wrong when you need a strong one). Some people just always write strong hypotheses. But it's easier to typo a strong hypothesis.

Robbie leaves a blank spot where the IH is, and fills it in after the step.

Practical Advice

How many base cases do you need?

Always at least one.

If you're analyzing recursive code or a recursive function, at least one for each base case of the code/function.

If you always go back s steps, at least s consecutive base cases.

Enough to make sure every case is handled.

Monochromatic Cows

Consider the following statements.

Suppose each cow is only one color.

Every group of cows contain only cows of the same color.

Spoof by strong induction:

$P(1)$: all groups of cows of size one contain only cows of one color.

(IH): $P(k) \rightarrow P(k + 1)$ for all $k \geq 1$.

Consider a group of $k + 1 \geq 2$ cows.

By our IH, all groups of size $\geq k$ contain cows of only one color.

So, break the $k + 1$ cows into 2 overlapping subgroups;

these by our IH are all the same color,

and since they share a cow in common the 2 groups are the same color.

Fill out the poll everywhere for
Activity Credit!

Go to pollev.com/cse311 and
login with your UW identity
Or text cse311 to 22333

Let's Try Another! Stamp Collecting

I have 4 cent stamps and 5 cent stamps (as many as I want of each).
Prove that I can make exactly n cents worth of stamps for all $n \geq 12$.

Try for a few values.

Then think...how would the inductive step go?

Stamp Collection (attempt)

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n .

Base Case:

12 cents can be made with three 4 cent stamps.

Inductive Hypothesis Suppose [maybe some other stuff and] $P(k)$, for an arbitrary $k \geq 12$.

Inductive Step:

We want to make $k + 1$ cents of stamps. By IH we can make $k - 3$ cents exactly with stamps. Adding another 4 cent stamp gives exactly $k + 1$ cents.

Stamp Collection

Is the proof right?

How do we know $P(13)$

We're not the base case, so our inductive hypothesis assumes $P(12)$, and then we say if $P(9)$ then $P(13)$.

Wait a second....

If you go back s steps every time, you need s base cases.

Or else the first few values aren't proven.

Stamp Collection

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n .

Base Case:

12 cents can be made with three 4 cent stamps.

13 cents can be made with two 4 cent stamps and one 5 cent stamp.

14 cents can be made with one 4 cent stamp and two 5 cent stamps.

15 cents can be made with three 5 cent stamps.

Inductive Hypothesis Suppose $P(12) \wedge P(13) \wedge \dots \wedge P(k)$, for an arbitrary $k \geq 15$.

Inductive Step:

We want to make $k + 1$ cents of stamps. By IH we can make $k - 3$ cents exactly with stamps. Adding another 4 cent stamp gives exactly $k + 1$ cents.

A good last check

After you've finished writing an inductive proof, pause.

If your inductive step always goes back s steps, you need s base cases (otherwise $b + 1$ will go back before the base cases you've shown). And make sure your inductive hypothesis is strong enough.

If your inductive step is going back a varying (unknown) number of steps, check the first few values above the base case, make sure your cases are really covered. And make sure your IH is strong.

Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n .
2. Base Cases: Show $P(b_{min}), P(b_{min+1}) \dots P(b_{max})$ i.e. show the base cases
3. Inductive Hypothesis: Suppose $P(b_{min}) \wedge P(b_{min} + 1) \wedge \dots \wedge P(k)$ for an arbitrary $k \geq b_{max}$. (The smallest value of k assumes all bases cases, but nothing else)
4. Inductive Step: Show $P(k + 1)$ (i.e. get $[P(b_{min}) \wedge \dots \wedge P(k)] \rightarrow P(k + 1)$)
5. Conclude by saying $P(n)$ is true for all $n \geq b_{min}$ by the principle of induction.

Stamp Collection, Done Wrong

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n .

Base Case:

12 cents can be made with three 4 cent stamps.

Inductive Hypothesis Suppose $P(k)$, $k \geq 12$.

Inductive Step:

We want to make $k + 1$ cents of stamps. By IH we can make k cents exactly with stamps. Replace one of the 4 cent stamps with a 5 cent stamp.

$P(n)$ holds for all n by the principle of induction.

Stamp Collection, Done Wrong

What if the starting point doesn't have any 4 cent stamps?

Like, say, 15 cents = $5+5+5$.