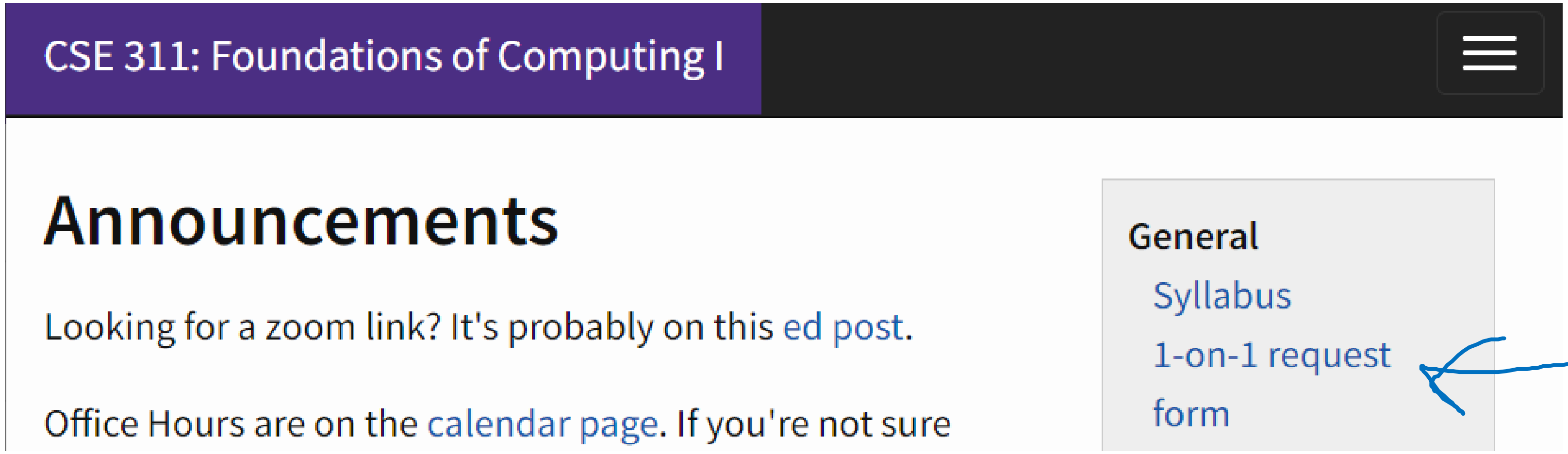
English Proofs and Sets

CSE 311 Spring 22
Lecture 9

Announcements

We have a new form on the homepage for “one-on-one” meetings



It’ll take us a few days to schedule, so not a good option for “I have a question about the current homework” but nice if you have a “topic X from last week never clicked, can we go back?”

Allie and Sandy’s office hour at 4:30 on Fridays will start with an hour of “how do we get started on a homework problem?” session.

Find The Bug

Let your domain of discourse be integers.
We claim that given $\forall x \exists y\ \texttt{Greater}(y,x)$, we can conclude $\exists y \forall x\ \texttt{Greater}(y,x)$
Where $\texttt{Greater}(y,x)$ means $y > x$

1.	$\forall x \exists y\ \texttt{Greater}(y,x)$	Given
2.	Let a be an arbitrary integer	--
3.	$\exists y\ \texttt{Greater}(y,a)$	Elim $\forall$ (1)
4.	$\texttt{Greater}(b,a)$	Elim $\exists$ (2)
5.	$\forall x\ \texttt{Greater}(b,x)$	Intro $\forall$ (4)
6.	$\exists y \forall x\ \texttt{Greater}(y,x)$	Intro $\exists$ (5)

$\forall x\ \text{Greater}(x+1, x)$

Find The Bug

1.	$\forall x \exists y$ `Greater`(y, x)	Given
2.	Let a be an arbitrary integer	--
3.	$\exists y$ `Greater`(y, a)	Elim $\forall$ (1)
4.	`Greater`(b, a)	Elim $\exists$ (2)
5.	$\forall x$ `Greater`(b, x)	Intro $\forall$ (4)
6.	$\exists y \forall x$ `Greater`(y, x)	Intro $\exists$ (5)

b is not a single number! The variable b depends on a. You can't get rid of a while b is still around.
What is b? It's probably something like $a + 1$.

Bug Found

There's one other "hidden" requirement to introduce $\forall$.

"No other variable in the statement can depend on the variable to be generalized"

Think of it like this: b was probably $a+1$ in that example.

You wouldn't have generalized from `Greater`$(a+1, a)$

To $\forall x$ `Greater`$(a+1, x)$. There's still an a, you'd have replaced all the a's.

x depends on y if y is in a statement when x is introduced.

This issue is much clearer in English proofs, which we'll start next time.

What's Next

We're taking off the training wheels!

Our goal with writing symbolic proofs was to prepare us to write proofs in English.

Let's get started.

The next 3 weeks:

- Practice communicating clear arguments to others.
- Learn new proof techniques.
- Learn fundamental objects (sets, number theory) that will let us talk more easily about computation at the end of the quarter.

Warm-up

Let your domain of discourse be integers.

Let $\texttt{Even}(x) := \exists y(x = 2y)$.

Prove "if x is even then x^2 is even."

Write a symbolic proof (with the extra rules "Definition of $\texttt{Even}$" and "Algebra").

Then we'll write it in English.

What's the claim in symbolic logic? $\forall x(\texttt{Even}(x) \rightarrow \texttt{Even}(x^2))$

Even

An integer x is even if (and only if) there exists an integer z, such that $x = 2z$.

Breakdown the statement

"if x is even then x^2 is even."

In symbols, that's: $\forall x \left(\text{Even}(x) \rightarrow \text{Even}(x^2)\right)$

Let's break down the statement to understand what the proof needs to look like:

$\forall x$ comes first. We need to introduce an arbitrary variable

$\text{Even}(x) \rightarrow \text{Even}(x^2)$ is left. We prove implications by assuming the hypothesis and setting the conclusion as our goal

$\text{Even}(x)$ is our starting assumption, $\text{Even}(x^2)$ is our goal

If x is even, then x^2 is even.

1. Let a be arbitrary

2.1 $\texttt{Even}(a)$ — Assumption

2.2 ? $\exists y\,(2y = a)$ — ? Defn of Even (2.1)

2.3 ? $2z = a$ — ? Elim $\exists$ (2.2)

2.4 ? $4z^2 = a^2$ — ?

2.5 ? $a^2 = 2\ 2z^2$ — ?

2.6 ? $\exists w\,(2w = a^2)$ — ?

2.7 $\texttt{Even}(a^2)$ — ?

3. $\texttt{Even}(a) \rightarrow \texttt{Even}(a^2)$ — Direct Proof Rule (2.1-2.7)

4. $\forall x(\texttt{Even}(x) \rightarrow \texttt{Even}(x^2))$ — Intro $\forall$ (3)

If x is even, then x^2 is even.

1. Let a be arbitrary	
2.1 $\mathtt{Even}(a)$	Assumption
2.2 $\exists y\ (2y = a)$	Definition of Even (2.1)
2.3 $2z = a$	Elim $\exists$(2.2)
2.4 $a^2 = 4z^2$	Algebra (2.3)
2.5 $a^2 = 2 \cdot 2z^2$	Alegbra (2.4)
2.6 $\exists w(2w = a^2)$	Intro $\exists$ (2.5)
2.7 $\mathtt{Even}(a^2)$	Definition of Even
3. $\mathtt{Even}(a) \rightarrow \mathtt{Even}(a^2)$	Direct Proof Rule (2.1-2.7)
4. $\forall x(\mathtt{Even}(x) \rightarrow \mathtt{Even}(x^2))$	Intro $\forall$ (3)

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4. $\forall x(\texttt{Even}(x) \rightarrow \texttt{Even}(x^2))$	Intro $\forall$ (3)

Let x be an arbitrary even integer.
By definition, there is an integer y such that $2y = x$.

Squaring both sides, we see that $x^2 = 4y^2 = 2 \cdot 2y^2$.

Because y is an integer, $2y^2$ is also an integer, and x^2 is two times an integer. Thus x^2 is even by the definition of even.

Since x was an arbitrary even integer, we can conclude that for every even x, x^2 is also even.

Converting to English

Start by introducing your assumptions.

Introduce variables with "let." Introduce assumptions with "suppose."

Always state what type your variable is. English proofs don't have an established domain of discourse.

Don't just use "algebra" explain what's going on.

We don't explicitly intro/elim ∃/∀ so we end up with fewer "dummy variables"

Let x be an arbitrary even integer.
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Let's do another!

First a definition

Rational

A real number x is rational if (and only if) there exist integers p and q, with $q \neq 0$ such that $x = p/q$.

$$\texttt{Rational}(x) \coloneqq \exists p \exists q(\,\texttt{Integer}(p) \land \texttt{Integer}(q) \land (x = p/q) \land q \neq 0)$$

Let’s do another!

“The product of two rational numbers is rational.”

What is this statement in predicate logic?

$\forall x \forall y([\mathtt{rational}(x) \wedge \mathtt{rational}(y)] \rightarrow \mathtt{rational}(xy))$

Remember unquantified variables in English are implicitly universally quantified.

$\forall x \forall y((\text{rational}(x) \wedge \text{rational}(y)) \rightarrow \text{rational}(xy))$

Doing a Proof

$\forall x \forall y([\texttt{rational}(x) \land \texttt{rational}(y)] \rightarrow \texttt{rational}(xy))$

"The product of two rational numbers is rational."

DON'T just jump right in!

Look at the statement, make sure you know:

1. What every word in the statement means.
2. What the statement as a whole means.
3. Where to start.
4. What your target is.

Let's do another!

"The product of two rational numbers is rational."

Let x, y be arbitrary rational numbers.

Therefore, xy is rational.

Since x and y were arbitrary, we can conclude the product of two rational numbers is rational.

Let's do another!

"The product of two rational numbers is rational."

Let x, y be arbitrary rational numbers.

By the definition of rational, $x = a/b$, $y = c/d$ for integers a, b, c, d where $b \neq 0$ and $d \neq 0$.

Multiplying, $xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Since integers are closed under multiplication, ac and bd are integers.

Moreover, $bd \neq 0$ because neither b nor d is 0. Thus xy is rational.

Since x and y were arbitrary, we can conclude the product of two rational numbers is rational.

Now You Try

$\forall x \forall y ([Even(x) \wedge Even(y)] \rightarrow Even(x+y))$

The sum of two even numbers is even.

1. Write the statement in predicate logic.
2. Write an English proof.
3. If you have lots of extra time, try writing the symbolic proof instead.

Let x, y be arbitrary even integers

Therefore x+y is also even

Now You Try

The sum of two even numbers is even.

Make sure you know:

1. What every word in the statement means.
2. What the statement as a whole means.
3. Where to start.
4. What your target is.

Even

An integer x is even if (and ~~only if~~) there exists an integer z, such that $x = 2z$.

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Help me adjust my explanation!

1. Write the statement in predicate logic.
2. Write an English proof.
3. If you have lots of extra time, try writing the symbolic proof instead.

Here's What I got.

$\forall x \forall y([\texttt{Even}(x) \wedge \texttt{Even}(y)] \rightarrow \texttt{Even}(x + y))$

Let x, y be arbitrary integers, and suppose x and y are even.

By the definition of even, $x = 2a, y = 2b$ for some integers a and b.

Summing the equations, $x + y = 2a + 2b = 2(a + b)$.

Since a and b are integers, $a + b$ is an integer, so $x + y$ is even by the definition of even.

Since x, y were arbitrary, we can conclude the sum of two even integers is even.

Why English Proofs?

Those symbolic proofs seemed pretty nice. Computers understand them, and can check them.

So what’s up with these English proofs?

They’re far easier for **people** to understand.

But instead of a computer checking them, now a human is checking them.

Sets

Sets

A set is an **unordered** group of **distinct** elements.

We'll always write a set as a list of its elements inside {curly, brackets}.

Variable names are capital letters, with lower-case letters for elements.

$|A| = 2$. "The size of A is 2." or "A has cardinality 2."

$A = \{\text{curly, brackets}\}$

$B = \{0,5,8,10\} = \{5,0,8,10\} = \{0,0,5,8,10\}$

$C = \{0,1,2,3,4, \dots\}$

Sets

Some more symbols:

$a \in A$ ("a is in A" or "a is an element of A") means a is one of the members of the set.

For $B = \{0,5,8,10\}, \;\; 0 \in B$.

$A \subseteq B$ (A is a subset of B) means every element of A is also in B.

For $A = \{1,2\}, B = \{1,2,3\}$ $A \subseteq B$

Sets

Be careful about these two operations:

If $A = \{1,2,3,4,5\}$

$\{1\} \subseteq A$, but $\{1\} \notin A$

$\in$ asks: is this item in that box?

$\subseteq$ asks: is everything in this box also in that box?

Try it!

Let $A = \{1,2,3,4,5\}$

$B = \{1,2,5\}$

Is $A \subseteq A$? Yes!

Is $B \subseteq A$? Yes

Is $A \subseteq B$? No

Is $\{1\} \in A$? No

Is $1 \in A$? Yes