Section 7

CSE 311 - Sp 2022

Administrivia

Announcements and Reminders

- HW5 and Midterm Grades will be released soon!
  - Regrade requests will be open like usual
  - If you are curious/concerned about your grade, set up a meeting with Robbie
- HW6
  - Due next Wednesday 5/18 @ 10pm
  - Lots of Induction (and one proof by Contradiction)
  - Start early so you have time to think and ask questions!

References

- How to LaTeX
  - https://courses.cs.washington.edu/courses/cse311/22sp/assignments/HowToLaTeX.pdf
- Logical Equivalences
  - https://courses.cs.washington.edu/courses/cse311/22sp/resources/reference-logical_equiv.pdf
- Inference Rules
  - https://courses.cs.washington.edu/courses/cse311/22sp/resources/InferenceRules.pdf
- Set Definitions
  - https://courses.cs.washington.edu/courses/cse311/22sp/resources/reference-sets.pdf
- Modular Arithmetic Definitions and Properties
  - https://courses.cs.washington.edu/courses/cse311/22sp/resources/reference-number-theory.pdf
- Induction Templates
  - https://courses.cs.washington.edu/courses/cse311/22sp/resources/induction-templates.pdf

Induction with Inequalities

Induction with Inequalities

- Induction with equalities and definitions like we’ve done so far can be more straightforward
- It can be hard to see the “magic fact” you need to substitute to complete the proof
- So, **scratch work is necessary**! (But you still need to write it up formally in your proof, scratch work is not sufficient evidence on its own)
- Also, make sure you know where you’re starting and where you’re going – it makes finding that “magic fact” easier!

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

What kind of induction should we use? Why?

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

What kind of induction should we use? Why?

Weak Induction!

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

What kind of induction should we use? Why?

Weak Induction!

Looking at the formula we're trying to prove, we only need to "go back one step." In other words, to prove P(k+1), we only need to know P(k). So, strong induction would be overkill here.

(note: it's not incorrect, you can do strong induction every time if you like, it's just more work imo)

Also, we're not doing induction on any kind of structure here (like a string or a tree), so structural induction probably wouldn't make much sense.

Weak Induction Template

Let ***P(n)*** be "(predicate you're trying to prove, must evaluate to a truth value)".
We show ***P(n)*** holds for (some range of) ***n*** by induction on ***n***.

<u>Base Case:</u> Show ***P(b)*** is true

<u>Inductive Hypothesis:</u> Suppose ***P(k)*** holds for an arbitrary $\boldsymbol{k \geq b}$

<u>Inductive Step:</u> Show ***P(k + 1)*** (i.e. get ***P(k) → P(k + 1)***)

<u>Conclusion:</u> Therefore, ***P(n)*** holds for all ***n*** by the principle of induction.

Weak Induction Template

Let ***P(n)*** be "(predicate you're trying to prove, must evaluate to a truth value)".
We show ***P(n)*** holds for (some range of) ***n*** by induction on ***n***.

Base Case: Show ***P(b)*** is true

Inductive Hypothesis: Suppose ***P(k)*** holds for an arbitrary ***k ≥ b***

Inductive Step: Show ***P(k + 1)*** (i.e. get ***P(k) → P(k + 1)***)

Conclusion: Therefore, ***P(n)*** holds for all ***n*** by the principle of induction.

The inductive step can be tricky with inequality! So make sure you know where you're starting and where you're going!

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let ***P(n)*** be "(predicate you're trying to prove, must evaluate to a truth value)".
We show ***P(n)*** holds for (some range of) ***n*** by induction on ***n***.

<u>Base Case:</u> Show ***P(b)*** is true

<u>Inductive Hypothesis:</u> Suppose ***P(k)*** holds for an arbitrary ***k ≥ b***

<u>Inductive Step:</u> Show ***P(k + 1)*** (i.e. get ***P(k) → P(k + 1)***)

<u>Conclusion:</u> Therefore, ***P(n)*** holds for all ***n*** by the principle of induction.

Work on this proof with the people around you, and then we'll go over it together!

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "".
We show P(n) holds for integers (in range) by induction on n.

<u>Base Case:</u>

<u>Inductive Hypothesis:</u> Suppose P(k) holds for an arbitrary k ≥ (base case), i.e. (IH in terms of P(n))

<u>Inductive Step:</u> Goal: show P(k+1): (IS goal in terms of P(n))

<u>Conclusion:</u> Therefore, P(n) holds for all integers (in range) by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers (in range) by induction on n.

Base Case:

Inductive Hypothesis: Suppose P(k) holds for an arbitrary k ≥ (base case), i.e. (IH in terms of P(n))

Inductive Step: Goal: show P(k+1): (IS goal in terms of P(n))

Conclusion: Therefore, P(n) holds for all integers (in range) by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

<u>Base Case:</u>

<u>Inductive Hypothesis:</u> Suppose P(k) holds for an arbitrary k ≥ (base case),
i.e. (IH in terms of P(n))

<u>Inductive Step:</u> Goal: show P(k+1): (IS goal in terms of P(n))

<u>Conclusion:</u> Therefore, P(n) holds for all integers (in range) by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

<u>Base Case:</u> P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

<u>Inductive Hypothesis:</u> Suppose P(k) holds for an arbitrary k ≥ (base case),
i.e. (IH in terms of P(n))

<u>Inductive Step:</u> Goal: show P(k+1): (IS goal in terms of P(n))

<u>Conclusion:</u> Therefore, P(n) holds for all integers (in range) by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

Base Case: P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. (IH in terms of P(n))

Inductive Step: Goal: show P(k+1): (IS goal in terms of P(n))

Conclusion: Therefore, P(n) holds for all integers (in range) by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

<u>Base Case:</u> P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

<u>Inductive Hypothesis:</u> Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. $6k + 6 < 2^k$

<u>Inductive Step:</u> Goal: show P(k+1): (IS goal in terms of P(n))

<u>Conclusion:</u> Therefore, P(n) holds for all integers (in range) by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

<u>Base Case:</u> P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

<u>Inductive Hypothesis:</u> Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. $6k + 6 < 2^k$

<u>Inductive Step:</u> Goal: show P(k+1): $6(k+1) + 6 < 2^{(k+1)}$

<u>Conclusion:</u> Therefore, P(n) holds for all integers (in range) by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

Base Case: P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. $6k + 6 < 2^k$

Inductive Step: Goal: show P(k+1): $6(k+1) + 6 < 2^{(k+1)}$

Conclusion: Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

<u>Base Case:</u> P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

<u>Inductive Hypothesis:</u> Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. $6k + 6 < 2^k$

<u>Inductive Step:</u> Goal: show P(k+1): $6(k+1) + 6 < 2^{(k+1)}$

Even if you get stuck here and can't figure out what to do in the IS, filling out the "proof skeleton" like this will get you more than half the points on an inductive proof! So focus on this skeleton first, and then see if you can apply some definitions for your IS that can make the left and right sides look more similar to complete the proof.

<u>Conclusion:</u> Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

Base Case: P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. $6k + 6 < 2^k$

Inductive Step: Goal: show P(k+1): $6(k+1) + 6 < 2^{(k+1)}$

$6(k+1) + 6 =$

Conclusion: Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

<u>Base Case:</u> P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

<u>Inductive Hypothesis:</u> Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. $6k + 6 < 2^k$

<u>Inductive Step:</u> Goal: show P(k+1): $6(k+1) + 6 < 2^{(k+1)}$

$$6(k+1) + 6 = 6k + 6 + 6$$

<u>Conclusion:</u> Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

<u>Base Case:</u> P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

<u>Inductive Hypothesis:</u> Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. $6k + 6 < 2^k$

<u>Inductive Step:</u> Goal: show P(k+1): $6(k+1) + 6 < 2^{(k+1)}$

$$6(k+1) + 6 = 6k + 6 + 6$$
$$< 2^k + 6 \quad \text{by Inductive Hypothesis}$$

<u>Conclusion:</u> Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

<u>Base Case:</u> P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

<u>Inductive Hypothesis:</u> Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. $6k + 6 < 2^k$

<u>Inductive Step:</u> Goal: show P(k+1): $6(k+1) + 6 < 2^{(k+1)}$

$6(k+1) + 6 = 6k + 6 + 6$

$< 2^k + 6$ by Inductive Hypothesis

$< 2^k + 2^k$ $2^k > 6$, since $k \geq 6$

<u>Conclusion:</u> Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

Base Case: P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. $6k + 6 < 2^k$

Inductive Step: Goal: show P(k+1): $6(k+1) + 6 < 2^{(k+1)}$

$6(k+1) + 6 = 6k + 6 + 6$

$< 2^k + 6$ — by Inductive Hypothesis

$< 2^k + 2^k$ — $2^k > 6$, since $k \geq 6$

$= 2 \cdot 2^k$

Conclusion: Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

Base Case: P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. $6k + 6 < 2^k$

Inductive Step: Goal: show P(k+1): $6(k+1) + 6 < 2^{(k+1)}$

$6(k+1) + 6 = 6k + 6 + 6$

$< 2^k + 6$ by Inductive Hypothesis

$< 2^k + 2^k$ $2^k > 6$, since $k \geq 6$

$= 2 \cdot 2^k$

$= 2^{(k+1)}$

Conclusion: Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Problem 1 - Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for integers $n \geq 6$ by induction on n.

<u>Base Case:</u> P(6): n=6, so $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P(6) holds.

<u>Inductive Hypothesis:</u> Suppose P(k) holds for an arbitrary $k \geq 6$,
i.e. $6k + 6 < 2^k$

<u>Inductive Step:</u> Goal: show P(k+1): $6(k+1) + 6 < 2^{(k+1)}$

$$
\begin{aligned}
6(k+1) + 6 &= 6k + 6 + 6 \\
&< 2^k + 6 && \text{by Inductive Hypothesis} \\
&< 2^k + 2^k && 2^k > 6, \text{ since } k \geq 6 \\
&= 2 \cdot 2^k \\
&= 2^{(k+1)}
\end{aligned}
$$

So, P(k + 1) holds!

<u>Conclusion:</u> Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Structural Induction

Structural Induction

- This can seem kind of confusing or weird, but really it's just an extension of the kinds of induction you've already used
- We can think of the natural numbers as a recursively defined set, so all the induction we've done is like a special case of structural induction
  - Basis Step: $0 \in \mathbb{N}$
  - Recursive Step: if $k \in \mathbb{N}$, then $k+1 \in \mathbb{N}$
- Often, the key is trying out small examples (e.g., writing out strings, drawing some trees, etc.)

Structural Induction Template (also on course website!)

For an $x \in S$, let *P(x)* be "(whatever you're trying to prove, must evaluate to a truth value)"
We show *P(x)* holds for all $x \in S$ by structural induction on x.

Base Case: Show $P(x)$ for all basis rules x in S

Inductive Hypothesis: Suppose $P(x)$ for all x listed as in S in the recursive rules.

Inductive Step: Show $P(?)$ holds for the "new element" given.

You will need a separate step for every rule!

Conclusion: Therefore $P(x)$ holds for all $x \in S$ by the principle of induction.

Problem 3b - Structural Induction on Trees

Definition of Tree:
Basis Step: • is a Tree.
Recursive Step: If L is a Tree and R is a Tree then Tree(•, L, R) is a Tree

Definition of leaves():
leaves(•) = 1
leaves(Tree(•, L, R)) = leaves(L) + leaves(R)

Definition of size():
size(•) = 1
size(Tree(•, L, R)) =1 + size(L) + size(R)

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

Work on this proof with the people around you, and then we'll go over it together!

Problem 3b - Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For x ∈ S, let P(x) be “”.
We show P(x) holds for all x ∈ S by structural induction on x.

Base Case: Show P(x) (for all x in the basis rules)

Inductive Hypothesis: Suppose P(x) (for all x in the recursive rules), i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(x) holds for all x ∈ S by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be “leaves(T) ≥ size(T)/2 + 1/2”.
We show P(x) holds for all x ∈ S by structural induction on x.

<u>Base Case:</u> Show P(x) (for all x in the basis rules)

<u>Inductive Hypothesis:</u> Suppose P(x) (for all x in the recursive rules),
i.e. (IH in terms of P(x))

<u>Inductive Step:</u> Goal: Show that P(?) holds. (IS goal in terms of P(?))

<u>Conclusion:</u> Therefore P(x) holds for all x ∈ S by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: Show P(x) (for all x in the basis rules)

Inductive Hypothesis: Suppose P(x) (for all x in the recursive rules),
i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(x) holds for all x ∈ S by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(x) (for all x in the recursive rules),
i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(x) holds for all x ∈ S by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(x) holds for all x ∈ S by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

<u>Base Case:</u> P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

<u>Inductive Hypothesis:</u> Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

<u>Inductive Step:</u> Goal: Show that P(?) holds. (IS goal in terms of P(?))

<u>Conclusion:</u> Therefore P(x) holds for all x ∈ S by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

<u>Base Case:</u> P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

<u>Inductive Hypothesis:</u> Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

<u>Inductive Step:</u> Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

<u>Conclusion:</u> Therefore P(x) holds for all x ∈ S by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

Again, as long as you can get this far, you will get the majority of points on the problem! Go for this skeleton first, and then think about what you need to do to complete the proof.

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

<u>Base Case:</u> P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

<u>Inductive Hypothesis:</u> Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

<u>Inductive Step:</u> Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R)) =

<u>Conclusion:</u> Therefore P(T) holds for all trees T by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R)) = leaves(L) + leaves(R) definition of leaves

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R))	= leaves(L) + leaves(R)	definition of leaves
	≥ (size(L)/2 + 1/2) + (size(R)/2 + 1/2)	by Inductive Hypothesis

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R))	= leaves(L) + leaves(R)	definition of leaves
	≥ (size(L)/2 + 1/2) + (size(R)/2 + 1/2)	by Inductive Hypothesis
	= (1/2 + size(L)/2 + size(R)/2) + 1/2	

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R))	= leaves(L) + leaves(R)	definition of leaves
	≥ (size(L)/2 + 1/2) + (size(R)/2 + 1/2)	by Inductive Hypothesis
	= (1/2 + size(L)/2 + size(R)/2) + 1/2	
	= (1 + size(L) + size(R)) / 2 + 1/2	

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R)) = leaves(L) + leaves(R)	definition of leaves
≥ (size(L)/2 + 1/2) + (size(R)/2 + 1/2)	by Inductive Hypothesis
= (1/2 + size(L)/2 + size(R)/2) + 1/2	
= (1 + size(L) + size(R)) / 2 + 1/2	
= size(Tree(•, L, R))/2 + 1/2	definition of size

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 3b - Structural Induction on Trees

Prove that
leaves(T) ≥ size(T)/2 + 1/2
for all Trees T

For a tree T, let P(T) be “leaves(T) ≥ size(T)/2 + 1/2”.
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R))	= leaves(L) + leaves(R)	definition of leaves
	≥ (size(L)/2 + 1/2) + (size(R)/2 + 1/2)	by Inductive Hypothesis
	= (1/2 + size(L)/2 + size(R)/2) + 1/2	
	= (1 + size(L) + size(R)) / 2 + 1/2	
	= size(Tree(•, L, R))/2 + 1/2	definition of size

So, P(Tree(•, L, R)) holds!

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Extra Induction
(if we have time)

Problem 3a - Structural Induction on Strings

Definition of string:
Basis Step: "" is a string.
Recursive Step: If X is a string and c is a character then append(c, X) is a string.

Definition of len():
len("") = 0
len(append(c, X)) = 1 + len(X)

Definition of double():
double("") = ""
double(append(c, X)) = append(c, append(c, double(X)))

Prove that for any string X, len(double(X)) = 2len(X).

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For x ∈ S, let P(x) be “”.
We show P(x) holds for all x ∈ S by structural induction on x.

Base Case: Show P(x) (for all x in the basis rules)

Inductive Hypothesis: Suppose P(x) (for all x in the recursive rules), i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(x) holds for all x ∈ S by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We show P(x) holds for all x ∈ S by structural induction on x.

Base Case: Show P(x) (for all x in the basis rules)

Inductive Hypothesis: Suppose P(x) (for all x in the recursive rules),
i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(x) holds for all x ∈ S by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We prove P(X) for all strings X by structural induction on X

Base Case: Show P(x) (for all x in the basis rules)

Inductive Hypothesis: Suppose P(x) (for all x in the recursive rules), i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(x) holds for all x ∈ S by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We prove P(X) for all strings X by structural induction on X

Base Case: P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

Inductive Hypothesis: Suppose P(x) (for all x in the recursive rules),
i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(x) holds for all x ∈ S by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be “len(double(X)) = 2len(X)”.
We prove P(X) for all strings X by structural induction on X

Base Case: P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

Inductive Hypothesis: Suppose P(X) holds for some arbitrary string X,
i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(x) holds for all x ∈ S by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We prove P(X) for all strings X by structural induction on X

Base Case: P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

Inductive Hypothesis: Suppose P(X) holds for some arbitrary string X,
i.e. len(double(X)) = 2len(X)

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(x) holds for all x ∈ S by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We prove P(X) for all strings X by structural induction on X

<u>Base Case:</u> P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

<u>Inductive Hypothesis:</u> Suppose P(X) holds for some arbitrary string X,
i.e. len(double(X)) = 2len(X)

<u>Inductive Step:</u> Goal: Show P(append(c, X)) for any c: len(double(append(c, X))) = 2(len(append(c, X)))

<u>Conclusion:</u> Therefore P(x) holds for all x ∈ S by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be “len(double(X)) = 2len(X)”.
We prove P(X) for all strings X by structural induction on X

Base Case: P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

Inductive Hypothesis: Suppose P(X) holds for some arbitrary string X,
i.e. len(double(X)) = 2len(X)

Inductive Step: Goal: Show P(append(c, X)) for any c: len(double(append(c, X))) = 2(len(append(c, X)))

Conclusion: Therefore P(X) holds for all strings X by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We prove P(X) for all strings X by structural induction on X

Base Case: P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

Inductive Hypothesis: Suppose P(X) holds for some arbitrary string X,
i.e. len(double(X)) = 2len(X)

Inductive Step: Goal: Show P(append(c, X)) for any c: len(double(append(c, X))) = 2(len(append(c, X)))

len(double(append(c, X))) =

Conclusion: Therefore P(X) holds for all strings X by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We prove P(X) for all strings X by structural induction on X

Base Case: P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

Inductive Hypothesis: Suppose P(X) holds for some arbitrary string X,
i.e. len(double(X)) = 2len(X)

Inductive Step: Goal: Show P(append(c, X)) for any c: len(double(append(c, X))) = 2(len(append(c, X)))

len(double(append(c, X))) = len(append(c, append(c, double(X)))) — definition of double

Conclusion: Therefore P(X) holds for all strings X by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We prove P(X) for all strings X by structural induction on X

Base Case: P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

Inductive Hypothesis: Suppose P(X) holds for some arbitrary string X,
i.e. len(double(X)) = 2len(X)

Inductive Step: Goal: Show P(append(c, X)) for any c: len(double(append(c, X))) = 2(len(append(c, X)))

len(double(append(c, X)))	= len(append(c, append(c, double(X))))	definition of double
	= 1 + len(append(c, double(X)))	definition of len

Conclusion: Therefore P(X) holds for all strings X by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be “len(double(X)) = 2len(X)”.
We prove P(X) for all strings X by structural induction on X

Base Case: P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

Inductive Hypothesis: Suppose P(X) holds for some arbitrary string X,
i.e. len(double(X)) = 2len(X)

Inductive Step: Goal: Show P(append(c, X)) for any c: len(double(append(c, X))) = 2(len(append(c, X)))

len(double(append(c, X)))	= len(append(c, append(c, double(X))))	definition of double
	= 1 + len(append(c, double(X)))	definition of len
	= 1 + 1 + len(double(X))	definition of len

Conclusion: Therefore P(X) holds for all strings X by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We prove P(X) for all strings X by structural induction on X

<u>Base Case:</u> P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

<u>Inductive Hypothesis:</u> Suppose P(X) holds for some arbitrary string X,
i.e. len(double(X)) = 2len(X)

<u>Inductive Step:</u> Goal: Show P(append(c, X)) for any c: len(double(append(c, X))) = 2(len(append(c, X)))

len(double(append(c, X)))	= len(append(c, append(c, double(X))))	definition of double
	= 1 + len(append(c, double(X)))	definition of len
	= 1 + 1 + len(double(X))	definition of len
	= 2 + 2len(X)	by I.H.

<u>Conclusion:</u> Therefore P(X) holds for all strings X by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We prove P(X) for all strings X by structural induction on X

Base Case: P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

Inductive Hypothesis: Suppose P(X) holds for some arbitrary string X,
i.e. len(double(X)) = 2len(X)

Inductive Step: Goal: Show P(append(c, X)) for any c: len(double(append(c, X))) = 2(len(append(c, X)))

len(double(append(c, X)))	= len(append(c, append(c, double(X))))	definition of double
	= 1 + len(append(c, double(X)))	definition of len
	= 1 + 1 + len(double(X))	definition of len
	= 2 + 2len(X)	by I.H.
	= 2(1 + len(X))	

Conclusion: Therefore P(X) holds for all strings X by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We prove P(X) for all strings X by structural induction on X

<u>Base Case:</u> P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

<u>Inductive Hypothesis:</u> Suppose P(X) holds for some arbitrary string X,
i.e. len(double(X)) = 2len(X)

<u>Inductive Step:</u> Goal: Show P(append(c, X)) for any c: len(double(append(c, X))) = 2(len(append(c, X)))

len(double(append(c, X)))	= len(append(c, append(c, double(X))))	definition of double
	= 1 + len(append(c, double(X)))	definition of len
	= 1 + 1 + len(double(X))	definition of len
	= 2 + 2len(X)	by I.H.
	= 2(1 + len(X))	
	= 2(len(append(c, X)))	definition of len

<u>Conclusion:</u> Therefore P(X) holds for all strings X by structural induction.

Problem 3a - Structural Induction on Strings

Prove that for any string X, len(double(X)) = 2len(X)

For a string X, let P(X) be "len(double(X)) = 2len(X)".
We prove P(X) for all strings X by structural induction on X

Base Case: P(""): By definition, len(double("")) = len("") = 0 = 2 · 0 = 2len(""), so P("") holds

Inductive Hypothesis: Suppose P(X) holds for some arbitrary string X,
i.e. len(double(X)) = 2len(X)

Inductive Step: Goal: Show P(append(c, X)) for any c: len(double(append(c, X))) = 2(len(append(c, X)))

len(double(append(c, X)))	= len(append(c, append(c, double(X))))	definition of double
	= 1 + len(append(c, double(X)))	definition of len
	= 1 + 1 + len(double(X))	definition of len
	= 2 + 2len(X)	by I.H.
	= 2(1 + len(X))	
	= 2(len(append(c, X)))	definition of len

So, P(append(c, X)) holds!

Conclusion: Therefore P(X) holds for all strings X by structural induction.

That’s All, Folks!

Any questions?