Wrap-up Number Theory

CSE 311 Autumn 2023
Lecture 15

Announcements

HW4 due tonight

HW5 comes out tonight as well.

HW5 is 1.5 weeks, not 1 week!

Due Wed. Nov. 8

We recommend you aim to finish “Part 1” by next Friday (number theory proofs and computations—about the length of a normal hw), and do “Part 2” (induction, topic for next week, two proofs) after that.

but there’s just one deadline. It’s all due Wed. Nov 8.

Plan For Today

We don’t expect you to fully absorb the new material today.

Our goals are:

1. See that number theory results can make code faster in unexpected ways.

2. See a bit of code analysis (a preview of 332).

3. Hopefully say “oh neat, I understand a *little bit* about how secure online communication works”

You should not expect to fully understand *anything* from today.

We’re going to skip a bunch of slides today. If you’re interested read them for fun. If not, then skip them!

GCD and LCM

Greatest Common Divisor

The Greatest Common Divisor of a and b (gcd(a,b)) is the largest integer c such that $c|a$ and $c|b$

Least Common Multiple

The Least Common Multiple of a and b (lcm(a,b)) is the smallest positive integer c such that $a|c$ and $b|c$.

Try a few values…

gcd(100,125)

gcd(17,49)

gcd(17,34)

gcd(13,0)

lcm(7,11)

lcm(6,10)

How do you calculate a gcd?

You could:

Find the prime factorization of each

Take all the common ones. E.g.

gcd(24,20)=gcd$(2^3 \cdot 3, 2^2 \cdot 5) = 2\hat{}\{\min(2{,}3)\} = 2\hat{}2 = 4$.

(lcm has a similar algorithm – take the maximum number of copies of everything)

But that’s….really expensive. Mystery finds gcd.

```
public int Mystery(int m, int n){
     if(m<n){
          int temp = m;
          m=n;
          n=temp;
     }
     while(n != 0) {
          int rem = m % n;
          m=n;
          n=temp;
     }
     return m;
}
```

GCD fact

If a and b are positive integers, then gcd(a,b) = gcd(b, a % b)

Why is this true? The proof isn’t easy, it’s at the end of this deck.

Why should you care?

So…what's it good for?

Remember everything we're learning contributes to us eventually understanding RSA. This is a key step in generating keys.

Suppose I want to solve $7x \equiv 3(mod\ n)$

Just multiply both sides by $\frac{1}{7}$…

Oh wait. We want a number to multiply by 7 to get 1.

What number can we pick?

The next two slides are going to get more abstract…we're listing out the facts we need to solve that equation.

Bézout's Theorem

Bézout's Theorem

If a and b are positive integers, then there exist integers s and t such that
gcd(a,b)$= sa + tb$

We're not going to prove this theorem…

But it turns out `Mystery` can be extended to find them.

You saw how to do that in section!

So…what's it good for?

Suppose I want to solve $7x \equiv 3(mod\ n)$

Just multiply both sides by $\frac{1}{7}$…

Oh wait. We want a number to multiply by 7 to get 1.

If the gcd(7,n) = 1

Then $s \cdot 7 + tn = 1$, so $7s - 1 = -tn$ i.e. $n|(7s - 1)$ so $7s \equiv 1(mod\ n)$.

So the s from Bézout's Theorem is what we should multiply by!

Ok…how am I supposed to find s, t?

It turns out that while you're calculating the gcd (using the Mystery algorithm), you can keep some extra information recorded, and end up with the s, t

This is called the "extended Euclidian algorithm"

Examples in these slides.

Try it

Solve the equation $7y \equiv 3(mod\ 26)$

What do we need to find?

The multiplicative inverse of $7(\mathrm{mod}\ 26)$

Finding the inverse…

gcd(26,7) = gcd(7, 26%7) = gcd(7,5)

= gcd(5, 7%5) = gcd(5,2)

= gcd(2, 5%2) = gcd(2, 1)

= gcd(1, 2%1) = gcd(1,0)= 1.

$$26 = 3 \cdot 7 + 5\ ;\ 5 = 26\ - 3 \cdot 7$$

$$7 =\ 5 \cdot 1 + 2\ ;\ \ 2 = 7\ -\quad 5 \cdot 1$$

$$5 =\ 2 \cdot 2 + 1\ ;\ \ 1 = 5\ -\quad 2 \cdot 2$$

$$\begin{gathered} 1 = 5\ - 2 \cdot 2 \\ = 5 - 2(7 - 5 \cdot 1) \\ = 3 \cdot 5 - 2 \cdot 7 \\ = 3 \cdot (26 - 3 \cdot 7) - 2 \cdot 7 \\ 3 \cdot 26 - 11 \cdot 7 \end{gathered}$$

-11 is a multiplicative inverse of 7 for (mod 26) arithmetic!
We'll write that as 15, since we're working mod 26.

Try it

Solve the equation $7y \equiv 3(mod\ 26)$

What do we need to find?

The multiplicative inverse of $7\ (mod\ 26)$. We found it’s 15.

$15 \cdot 7 \cdot y \equiv 15 \cdot 3(mod\ 26)$

$y \equiv 45(mod\ 26)$

Or $y \equiv 19(mod\ 26)$

So $26|19 - y$, i.e. $26k = 19 - y$ (for $k \in \mathbb{Z}$) i.e. $y = 19 - 26 \cdot k$ for any $k \in \mathbb{Z}$

Solutions: $\{\dots, -7, 19, 45, \dots 19 + 26k, \dots\}$ i.e. $\{x: x = 19 + 26k$ for some $k \in \mathbb{Z}\}$

Proving the key fact about gcds

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \mathrm{gcd}(a, b)$ and $y = \mathrm{gcd}(b, a\%b)$.

We show that y is a common divisor of a and b.

By definition of gcd, $y|b$ and $y|(a\%b)$. So it is enough to show that $y|a$.

Applying the definition of divides we get $b = yk$ for an integer k, and $(a\%b) = yj$ for an integer j.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q.

Plugging in both of our other equations:

$a = qyk + yj = y(qk + j)$. Since q, k, and j are integers, $y|a$. Thus y is a common divisor of a, b and thus $y \leq x$.

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

By definition of gcd, $\mathrm{x}|b$ and $x|a$. So it is enough to show that $\mathrm{x}|(a\%b)$.

Applying the definition of divides we get $b = xk'$ for an integer k', and $\mathrm{a} = xj'$ for an integer j'.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q

Plugging in both of our other equations:

$xj' = qxk' + a\%b$. Solving for $a\%b$, we have $a\%b = xj' - qxk' = x(j' - qk')$. So $x|(a\%b)$. Thus x is a common divisor of $b, a\%b$ and thus $x \leq y$.

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

We have shown $x \leq y$ and $y \leq x$.

Thus $x = y$, and $\gcd(a, b) = \gcd(b, a\%b)$.

Euclidian Algorithm

Euclid’s Algorithm

```
while(n != 0) {
             int rem = m % n;
             m=n;
             n=rem;
       }
```

gcd(660,126)

Euclid’s Algorithm

```
while(n != 0) {
             int rem = m % n;
             m=n;
             n=rem;
      }
```

$$\begin{aligned}
\gcd(660,126) &= \gcd(126, 660 \bmod 126) &&= \gcd(126, 30)\\
&= \gcd(30, 126 \bmod 30) &&= \gcd(30, 6)\\
&= \gcd(6, 30 \bmod 6) &&= \gcd(6, 0)\\
&= 6
\end{aligned}$$

Tableau form

$$\begin{aligned}
\boxed{660} &= 5 \cdot \boxed{126} + 30\\
126 &= 4 \cdot 30 + \enclose{circle}{6}\\
30 &= 5 \cdot 6 + 0
\end{aligned}$$

Starting Numbers

Final answer

Bézout's Theorem

Bézout's Theorem

If a and b are positive integers, then there exist integers s and t such that
gcd(a,b)$= sa + tb$

We're not going to prove this theorem…

But we'll show you how to find s,t for any positive integers a, b.

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

gcd(35,27)

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

gcd(35,27) = gcd(27, 35%27) = gcd(27,8)

= gcd(8, 27%8) = gcd(8, 3)

= gcd(3, 8%3) = gcd(3, 2)

= gcd(2, 3%2) = gcd(2,1)

= gcd(1, 2%1) = gcd(1,0)

$$\boxed{\begin{aligned} 35 &= 1 \cdot 27 + 8 \\ 27 &= 3 \cdot 8 + 3 \\ 8 &= 2 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \end{aligned}}$$

Extended Euclidian Algorithm

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Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

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$$\boxed{\begin{aligned} 8 &= 35 - 1 \cdot 27 \\ 3 &= 27 - 3 \cdot 8 \\ 2 &= 8 - 2 \cdot 3 \\ 1 &= 3 - 1 \cdot 2 \end{aligned}}$$

$$\begin{aligned} 1 &= 3 - 1 \cdot 2 \\ &= 3 - 1 \cdot (8 - 2 \cdot 3) \\ &= -1 \cdot 8 + 2 \cdot 3 \end{aligned}$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$\boxed{\begin{aligned} 8 &= 35 - 1 \cdot 27 \\ 3 &= 27 - 3 \cdot 8 \\ 2 &= 8 - 2 \cdot 3 \\ 1 &= 3 - 1 \cdot 2 \end{aligned}}$$

$$\begin{aligned} 1 &= 3 - 1 \cdot 2 \\ &= 3 - 1 \cdot (8 - 2 \cdot 3) \\ &= -1 \cdot 8 + 3 \cdot 3 \\ &= -1 \cdot 8 + 3(27 - 3 \cdot 8) \\ &= 3 \cdot 27 - 10 \cdot 8 \\ &= 3 \cdot 27 - 10(35 - 1 \cdot 27) \\ &= 13 \cdot 27 - 10 \cdot 35 \end{aligned}$$

When substituting back, you keep the larger of m, n and the number you just substituted. Don't simplify further! (or you lose the form you need)

gcd(27,35) = $13 \cdot 27 + (-10) \cdot 35$

RSA Encryption

Key Steps in RSA

Given two numbers, we can find their gcd quickly.

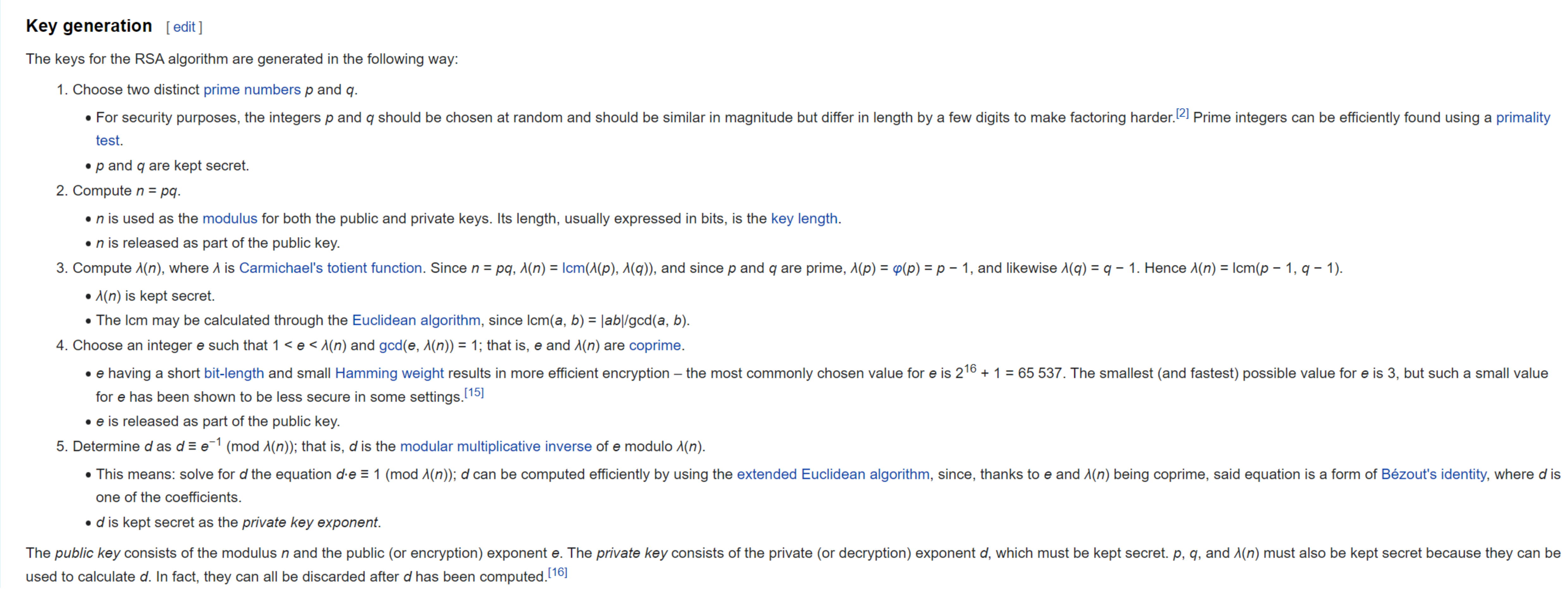
If we have an equation

$ax \equiv b \pmod{n}$

And $\gcd(a, n) = 1$ then we can quickly find a number to multiply the equation by to solve for x.

Framing Device

We’re going to give you enough background to (mostly) understand the RSA encryption system.



Key generation [edit]

The keys for the RSA algorithm are generated in the following way:

1. Choose two distinct prime numbers p and q.
   - For security purposes, the integers p and q should be chosen at random and should be similar in magnitude but differ in length by a few digits to make factoring harder.[2] Prime integers can be efficiently found using a primality test.
   - p and q are kept secret.
2. Compute $n = pq$.
   - n is used as the modulus for both the public and private keys. Its length, usually expressed in bits, is the key length.
   - n is released as part of the public key.
3. Compute $\lambda(n)$, where λ is Carmichael's totient function. Since $n = pq$, $\lambda(n) = \operatorname{lcm}(\lambda(p), \lambda(q))$, and since p and q are prime, $\lambda(p) = \varphi(p) = p - 1$, and likewise $\lambda(q) = q - 1$. Hence $\lambda(n) = \operatorname{lcm}(p - 1, q - 1)$.
   - $\lambda(n)$ is kept secret.
   - The lcm may be calculated through the Euclidean algorithm, since $\operatorname{lcm}(a, b) = |ab|/\gcd(a, b)$.
4. Choose an integer e such that $1 < e < \lambda(n)$ and $\gcd(e, \lambda(n)) = 1$; that is, e and $\lambda(n)$ are coprime.
   - e having a short bit-length and small Hamming weight results in more efficient encryption – the most commonly chosen value for e is $2^{16} + 1 = 65\,537$. The smallest (and fastest) possible value for e is 3, but such a small value for e has been shown to be less secure in some settings.[15]
   - e is released as part of the public key.
5. Determine d as $d \equiv e^{-1} \pmod{\lambda(n)}$; that is, d is the modular multiplicative inverse of e modulo $\lambda(n)$.
   - This means: solve for d the equation $d \cdot e \equiv 1 \pmod{\lambda(n)}$; d can be computed efficiently by using the extended Euclidean algorithm, since, thanks to e and $\lambda(n)$ being coprime, said equation is a form of Bézout's identity, where d is one of the coefficients.
   - d is kept secret as the *private key exponent*.

The *public key* consists of the modulus n and the public (or encryption) exponent e. The *private key* consists of the private (or decryption) exponent d, which must be kept secret. p, q, and $\lambda(n)$ must also be kept secret because they can be used to calculate d. In fact, they can all be discarded after d has been computed.[16]

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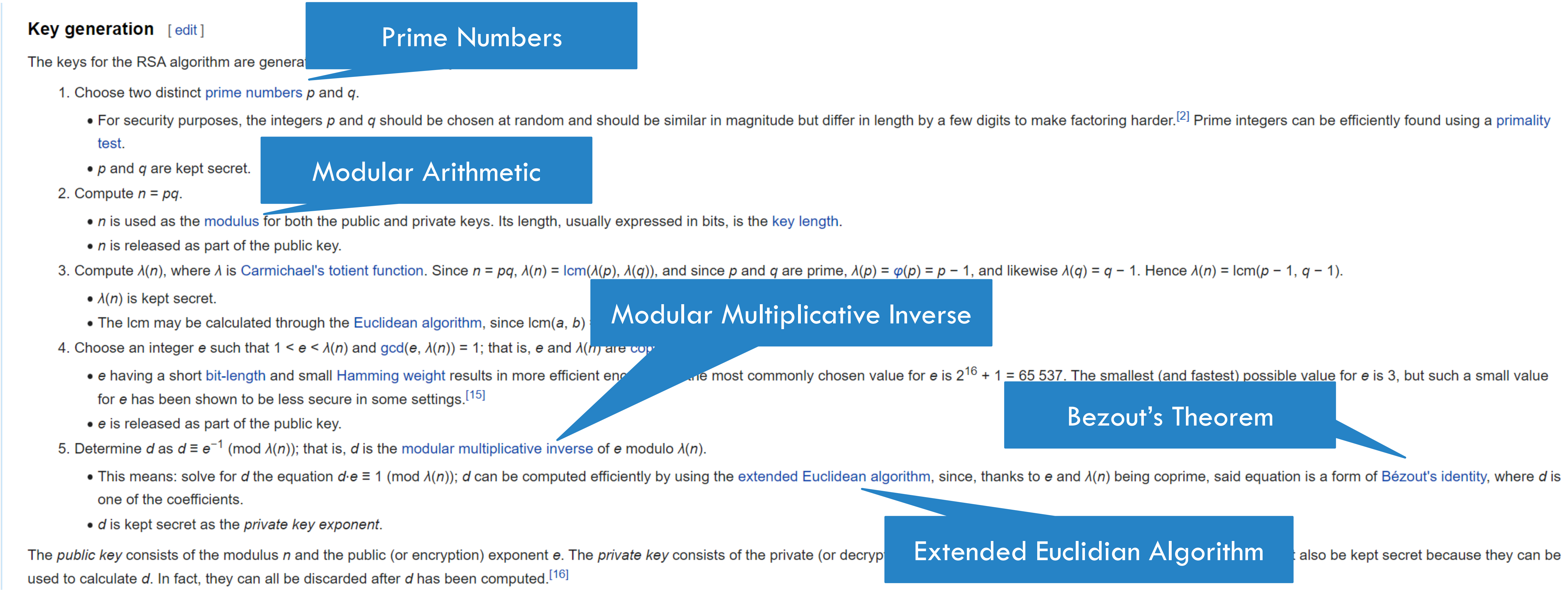

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Prime Numbers

Modular Arithmetic

Modular Multiplicative Inverse

Bezout's Theorem

Extended Euclidian Algorithm

Framing Device

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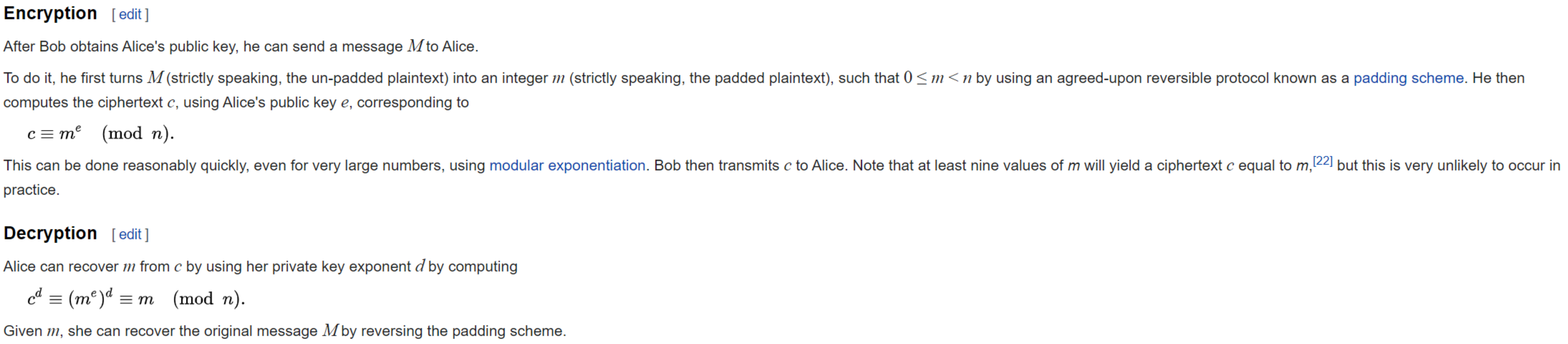

Encryption [edit]

After Bob obtains Alice's public key, he can send a message M to Alice.

To do it, he first turns M (strictly speaking, the un-padded plaintext) into an integer m (strictly speaking, the padded plaintext), such that $0 \le m < n$ by using an agreed-upon reversible protocol known as a padding scheme. He then computes the ciphertext c, using Alice's public key e, corresponding to

$$c \equiv m^e \pmod{n}.$$

This can be done reasonably quickly, even for very large numbers, using modular exponentiation. Bob then transmits c to Alice. Note that at least nine values of *m* will yield a ciphertext c equal to *m*,[22] but this is very unlikely to occur in practice.

Decryption [edit]

Alice can recover m from c by using her private key exponent d by computing

$$c^d \equiv (m^e)^d \equiv m \pmod{n}.$$

Given m, she can recover the original message M by reversing the padding scheme.

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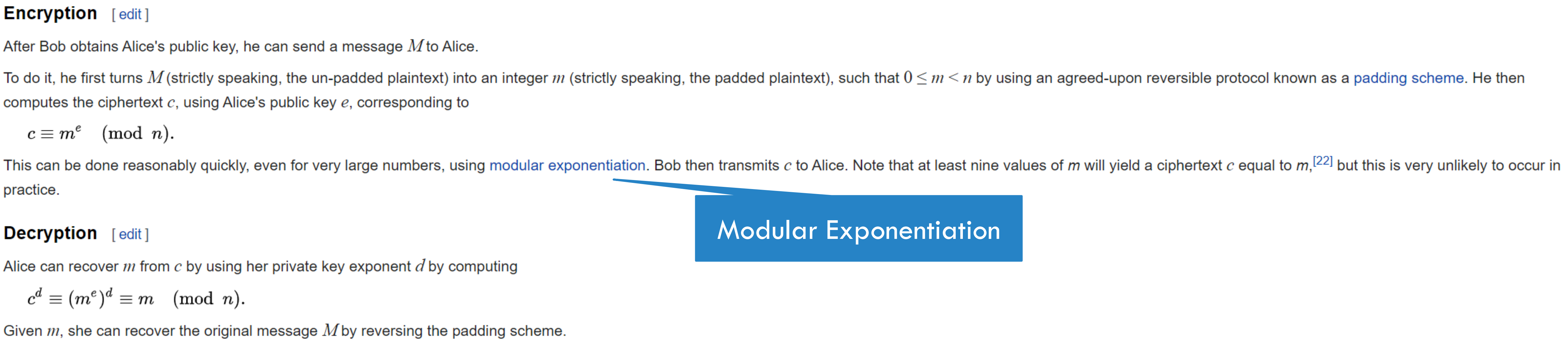

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Modular Exponentiation

Decryption [edit]

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$$c^d \equiv (m^e)^d \equiv m \pmod{n}.$$

Given m, she can recover the original message M by reversing the padding scheme.

An application of all of this modular arithmetic

Amazon chooses random 512-bit (or 1024-bit) prime numbers p, q and an exponent e (often about 60,000).

Amazon calculates $\mathrm{n} = pq$. They tell your computer (n, e) (not p, q)

You want to send Amazon your credit card number a.

You compute $C = a^e \% n$ and send Amazon C.

Amazon computes d, the multiplicative inverse of e $(mod\ [p - 1][q - 1])$

Amazon finds $C^d \% n$

Fact: $a = C^d \% n$ as long as $0 < a < n$ and $p \nmid a$ and $q \nmid a$

How big are those numbers?

1230186684530117755130494958384962720772853569595334792197322
4521517264005072636575187452021997864693899564749427740638459
2519255732630345373154826850791702612214291346167042921431160
22212404792747377940806653514195974598569021434l3

=

3347807169895689878604416984821269081770479498371376856891243
138898288379387800228761471165253174308773781446799948 9

×

3674604366679959042824463379962795263227915816434308764267603
22838157396665112792333734171433968102700927987363089l7

How do we accomplish those steps?

That fact? You can prove it in the extra credit problem on HW5. It's a nice combination of lots of things we've done with modular arithmetic.

Let's talk about finding $C = a^e \% n$.

e is a BIG number (about 2^{16} is a common choice)

```
int total = 1;
for(int i = 0; i < e; i++){
    total = (a * total) % n;
}
```

Fast Exponentiation Algorithm

Let’s build a faster algorithm.

Fast exponentiation – simple case. What if e is exactly 2^{16}?

```
int total = 1;
for(int i = 0; i < e; i++){
    total = a * total % n;
}
Instead:
int total = a;
for(int i = 0; i < log(e); i++){
    total = total^2 % n;
}
```

Fast Exponentiation Algorithm

What if e isn't exactly a power of 2?

Step 1: Write e in binary.

Step 2: Find $a^c \% n$ for c every power of 2 up to e.

Step 3: calculate a^e by multiplying a^c for all c where binary expansion of e had a 1.

Fast Exponentiation Algorithm

Find $4^{11}\%10$

Step 1: Write $\boldsymbol{e}$ in binary.

Step 2: Find $a^c\%n$ for c every power of 2 up to e.

Step 3: calculate a^e by multiplying a^c for all c where binary expansion of e had a 1.

Start with largest power of 2 less than e (8). 8's place gets a 1. Subtract power

Go to next lower power of 2, if remainder of e is larger, place gets a 1, subtract power; else place gets a 0 (leave remainder alone).

11 = 1011_2

Fast Exponentiation Algorithm

Find $4^{11}\%10$

Step 1: Write e in binary.

Step 2: Find $\boldsymbol{a^c\%n}$ for $\boldsymbol{c}$ every power of 2 up to $\boldsymbol{e}$.

Step 3: calculate a^e by multiplying a^c for all c where binary expansion of e had a 1.

$4^1\%10 = 4$

$4^2\%10 = 6$

$4^4\%10 = 6^2\%10 = 6$

$4^8\%10 = 6^2\%10 = 6$

Fast Exponentiation Algorithm

Find $4^{11}\%10$

Step 1: Write e in binary.

Step 2: Find $a^c\%n$ for c every power of 2 up to e.

Step 3: calculate $\boldsymbol{a^e}$ by multiplying $\boldsymbol{a^c}$ for all $\boldsymbol{c}$ where binary expansion of $\boldsymbol{e}$ had a $\mathbf{1}$.

$4^1\%10 = 4$

$4^2\%10 = 6$

$4^4\%10 = 6^2\%10 = 6$

$4^8\%10 = 6^2\%10 = 6$

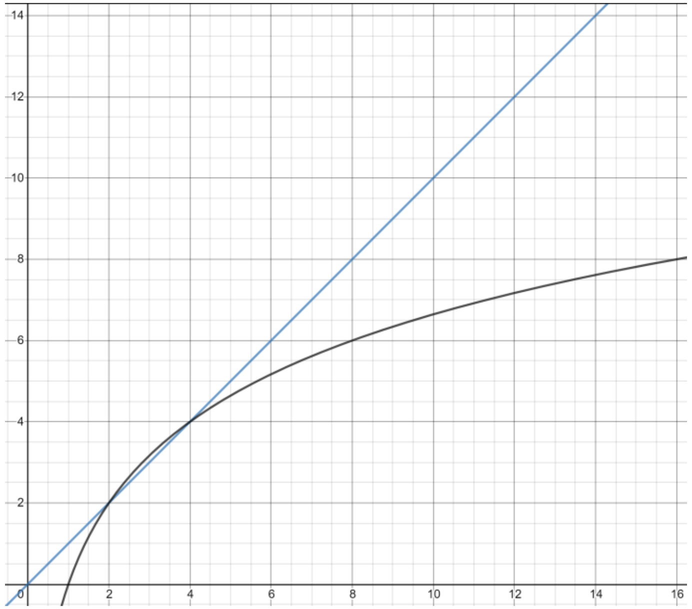
$$4^{11}\%10 = 4^{8+2+1}\%10 =$$
$$[(4^8\%10) \cdot \left(4^2\%10\right) \cdot (4\%10)]\%10 = (6 \cdot 6 \cdot 4)\%10$$
$$= (36\%10 \cdot 4)\%10 = (6 \cdot 4)\%10 = \ 24\%10 = 4.$$

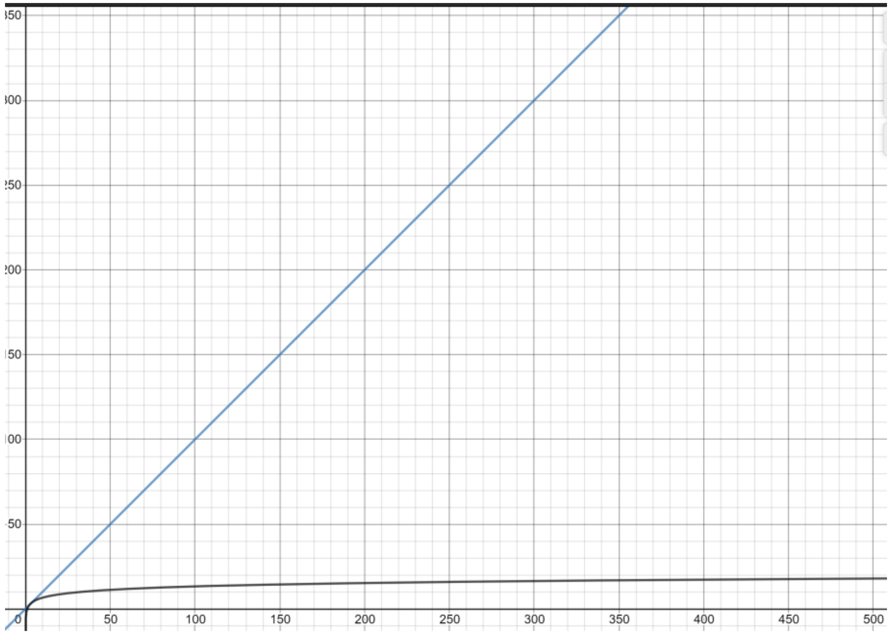
Fast Exponentiation Algorithm

Is it…actually fast?

The number of multiplications is between $\log_2 e$ and $2\log_2 e$.

That’s A LOT smaller than e





One More Example for Reference

Find $3^{25}\%7$ using the fast exponentiation algorithm.

Find 25 in binary:

16 is the largest power of 2 smaller than 25. $(25 - 16) = 9$ remaining

8 is smaller than 9. $(9 - 8) = 1$ remaining.

4s place gets a $\mathbf{0}$.

2s place gets a $\mathbf{0}$

$1s$ place gets a 1

11001_2

One More Example for Reference

Find $3^{25}\%7$ using the fast exponentiation algorithm.

Find $3^{2^i}\%7$:

$3^1\%7 = 3$

$3^2\%7 = 9\%7 = 2$

$3^4\%7 = (3^2 \cdot 3^2)\%7 = (2 \cdot 2)\%7 = 4$

$3^8\%7 = \left(3^4 \cdot 3^4\right)\%7 = (4 \cdot 4)\%7 = 2$

$3^{16}\%7 = \left(3^8 \cdot 3^8\right)\%7 = (2 \cdot 2)\%7 = 4$

One More Example for Reference

Find $3^{25}\%7$ using the fast exponentiation algorithm.

$3^1\%7 = 3$

$3^2\%7 = 2$

$3^4\%7 = 4$

$3^8\%7 = 2$

$3^{16}\%7 = 4$

$$\begin{aligned}
&3^{25}\%7 = 3^{16+8+1}\%7 \\
&= [(3^{16}\%7) \cdot \left(3^8\%7\right) \cdot (3^1\%7)]\%7 \\
&= [4 \cdot 2 \cdot 3]\%7 \\
&= (1 \cdot 3)\%7 = 3
\end{aligned}$$

A Brief Concluding Remark

Why does RSA work? i.e. why is my credit card number “secret”?

Raising numbers to large exponents (in mod arithmetic) and finding multiplicative inverses in modular arithmetic are things computers can do quickly.

But factoring numbers (to find p, q to get d) or finding an “exponential inverse” (not the real term) directly are not things computers can do quickly. At least as far as we know.

An application of all of this modular arithmetic

Amazon chooses random 512-bit (or 1024-bit) prime numbers p, q and an exponent e (often about 60,000).

Amazon calculates $\mathrm{n} = pq$. They tell your computer (n, e) (not p, q)

You want to send Amazon your credit card number a.

You compute $C = a^e \% n$ and send Amazon C.

Amazon computes d, the multiplicative inverse of e $(mod\ [p-1][q-1])$

Amazon finds $C^d \% n$

Fact: $a = C^d \% n$ as long as $0 < a < n$ and $p \nmid a$ and $q \nmid a$

If we have time…

We skipped some proofs…

…you've had a lot of good questions!

We missed:

A proof where we got stuck: main message, work from both ends! Be willing to erase things and try again!

Lecture 12, Slide 34

More practice with proof by contradiction.

Lecture 13, Slide 24

More practice with number theory definitions

End of this slide deck.

If we have time we'll practice them now. Otherwise we'll leave you to try them on your own.

Extra Practice!

Warm up

Equivalence in modular arithmetic

Let $a \in \mathbb{Z}, b \in \mathbb{Z}, n \in \mathbb{Z}$ and $n > 0$.
We say $a \equiv b\ (mod\ n)$ if and only if $n|(b - a)$

Show that $a \equiv b\ (mod\ n)$ if and only if $b \equiv a(mod\ n)$

Show that $a\%n$=$(a - n)\%n$ Where $b\%c$ is the unique r such that $b = kc + r$ for some integer k.

The Division Theorem

For every $a \in \mathbb{Z}$, $\boldsymbol{d} \in \mathbb{Z}$ with $\boldsymbol{d} > \boldsymbol{0}$
There exist *unique* integers q, r with $0 \leq r < d$ Such that $a = dq + r$

Warm up

Show that $a \equiv b\ (mod\ n)$ if and only if $b \equiv a(mod\ n)$

$a \equiv b(mod\ n) \leftrightarrow n|(b-a) \leftrightarrow nk = b - a(\text{for } k \in \mathbb{Z}) \leftrightarrow$

$n(-k) = a - b(\text{for } -\text{k} \in \mathbb{Z}) \leftrightarrow n|(a-b) \leftrightarrow b \equiv a(mod\ n)$

Show that $a\%n$=$(a-n)\%n$ Where $b\%c$ is the unique r such that $b = kc + r$ for some integer k.

By definition of %, $a = qn + (a\%n)$ for some integer q. Subtracting n,

$a - n = (q-1)n + (a\%n)$. Observe that $q-1$ is an integer, and that this is the form of the division theorem for $(a-n)\%n$. Since the division theorem guarantees a unique integer, $(a-n)\%n = (a\%n)$

% and Mod

Other resources use mod to mean an operation (takes in an integer, outputs an integer). We will not in this course. mod only describes $\equiv$. It's not "just on the right hand side"

Define $a\%b$ to be "the r you get from the division theorem"
i.e. the integer r such that $0 \leq r < d$ and $a = bq + r$ for some integer q.

This is the "mod function"

I claim $a\%n = b\%n$ if and only if $a \equiv b (mod\ n)$.

How do we show and if-and-only-if?

$a\%n = b\%n$ if and only if $a \equiv b (mod\ n)$

Backward direction:

Suppose $a \equiv b(mod\ n)$

$a\%n = (b - nk)\%n = b\%n$

$a\%n = b\%n$ if and only if $a \equiv b (mod\ n)$

Backward direction:

Suppose $a \equiv b(mod\ n)$

$n|b-a$ so $nk = b-a$ for some integer k. (by definitions of mod and divides).

So $a = b - nk$

Taking each side $\%n$ we get:

$a\%n = (b-nk)\%n = b\%n$

Where the last equality follows from k being an integer and doing k applications of the identity we proved in the warm-up.

$a\%n = b\%n$ if and only if $a \equiv b(mod\ n)$

Show the forward direction:

If $a\%n = b\%n$ then $a \equiv b(mod\ n)$.

This proof is a bit different than the other direction.

Remember to work from top and bottom!!

Equivalence in modular arithmetic

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Forward direction:

Suppose $a\%n = b\%n$.

By definition of %, $a = kn + (a\%n)$ and $b = jn + (b\%n)$ for integers k, j

Isolating $a\%n$ we have $a\%n = a - kn$. Since $a\%n = b\%n$, we can plug into the second equation to get: $b = jn + (a - kn)$

Rearranging, we have $b - a = (j - k)n$. Since k, j are integers we have $n|(b - a)$.

By definition of mod we have $a \equiv b(mod\ n)$.

More Number Theory Proofs

Caution

To fit proofs on these slides, I skipped some of the boilerplate steps (e.g. introducing variables as arbitrary, including a conclusion)

Don’t skip those on your homework/midterm, please ☺

Warm up

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Modular arithmetic so far

For all integers a, b, c, d, n where $n > 0$:

If $a \equiv b (mod\ n)$ then $a + c \equiv a + c (mod\ n)$.

If $a \equiv b (mod\ n)$ then $ac \equiv bc\ (mod\ n)$.

$a \equiv b (mod\ n)$ if and only if $b \equiv a\ (mod\ n)$.

$a\%n = (a - n)\%n$.

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Why does the Euclidian Algorithm Work?

Correctness of an algorithm

The key to the Euclidian Algorithm being correct is that each time through the loop, you don’t change the gcd of the variables `m,n`.

To prove the code correct, you really want an induction proof (it’s good practice to think about it!). The inductive step relies on the fact we stated but didn’t prove:

```
gcd(a,b) = gcd(b, a%b).
```

Let’s prove it!

GCD fact

If a and b are positive integers, then gcd(a,b) = gcd(b, a % b)

How do you show two gcds are equal?

Call $a = \text{gcd}(w, x), b = \text{gcd}(y, z)$

If $b|w$ and $b|x$ then b is a common divisor of w, x so $b \leq a$

If $a|y$ and $a|z$ then a is a common divisor of $y, z,$ so $a \leq b$

If $a \leq b$ and $b \leq a$ then $a = b$

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \mathrm{gcd}(a, b)$ and $y = \mathrm{gcd}(b, a\%b)$.

We show that y is a common divisor of a and b.

By definition of gcd, $y|b$ and $y|(a\%b)$. So it is enough to show that $y|a$.

Applying the definition of divides we get $b = yk$ for an integer k, and $(a\%b) = yj$ for an integer j.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q.

Plugging in both of our other equations:

$a = qyk + yj = y(qk + j)$. Since q, k, and j are integers, $y|a$. Thus y is a common divisor of a, b and thus $y \leq x$.

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \mathrm{gcd}(a, b)$ and $y = \mathrm{gcd}(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

By definition of gcd, $\mathrm{x}|b$ and $x|a$. So it is enough to show that $\mathrm{x}|(a\%b)$.

Applying the definition of divides we get $b = xk'$ for an integer k', and $\mathrm{a} = xj'$ for an integer j'.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q

Plugging in both of our other equations:

$xj' = qxk' + a\%b$. Solving for $a\%b$, we have $a\%b = xj' - qxk' = x(j' - qk')$. So $x|(a\%b)$. Thus x is a common divisor of $b, a\%b$ and thus $x \leq y$.

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \mathrm{gcd}(a, b)$ and $y = \mathrm{gcd}(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

We have shown $x \leq y$ and $y \leq x$.

Thus $x = y$, and $\mathrm{gcd}(a, b) = \mathrm{gcd}(b, a\%b)$.