Strong Induction

CSE 311 Autumn 2023
Lecture 16

How do we know recursion works?

```
//Assume i is a nonnegative integer
//returns 2^i.
public int CalculatesTwoToTheI(int i){
     if(i == 0)
          return 1;
     else
          return 2*CaclulatesTwoToTheI(i-1);
}
```

Why does `CalculatesTwoToTheI(4)` calculate `2^4?`

`Convince the other people in your room`

Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n.
2. Base Case: Show $P(0)$ i.e. show the base case
3. Inductive Hypothesis: Suppose $P(k)$ for an arbitrary k.
4. Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)
5. Conclude by saying $P(n)$ is true for all n by the principle of induction.

The Principle of Induction (formally)

Principle of Induction

$$\frac{P(0);\forall k(P(k) \rightarrow P(k+1))}{\therefore \qquad \forall n(P(n))}$$

Informally: if you knock over one domino, and every domino knocks over the next one, then all your dominoes fell over.

More Induction

Induction doesn't **only** work for code!

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

More Induction

Induction doesn't **only** work for code!

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$."

We show $P(n)$ holds for all n by induction on n.

Base Case (   )

Inductive Hypothesis:

Inductive Step:

$P(n)$ holds for all $n \geq 0$ by the principle of induction.

More Induction

Induction doesn't **only** work for code!

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$."

We show $P(n)$ holds for all n by induction on n.

Base Case $(n = 0)$ $\sum_{i=0}^{0} 2^i = 1 = 2 - 1 = 2^{0+1} - 1$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

Inductive Step: We show $P(k+1)$. Consider the summation $\sum_{i=0}^{k+1} 2^i = 2^{\mathrm{k}+1} + \sum_{i=0}^{k} 2^i = 2^{k+1} + 2^{k+1} - 1$, where the last step is by IH.

Simplifying, we get: $\sum_{i=0}^{k+1} 2^i = 2^{k+1} + 2^{k+1} - 1 = 2 \cdot 2^{k+1} - 1 = 2^{(k+1)+1} - 1$.

$P(n)$ holds for all $n \geq 0$ by the principle of induction.

Let's Try Another Induction Proof

Let $g(n) = \begin{cases} 2 & \text{if } n = 2 \\ g(n-1)^2 + 3g(n-1) & \text{if } n > 2 \end{cases}$

Prove $g(n)$ is even for all $n \geq 2$ by induction on n.

Let's just set this one up, we'll leave the individual pieces as exercises.

Setup

Let $P(n)$ be "$g(n)$ is even."

HEY WAIT -- $P(0)$ isn't true $g(0)$ isn't even defined!

We can move the "starting line"

Change the base case, and then update the IH to have the smallest value of k assume just the base case.

Setup

Let $P(n)$ be "$g(n)$ is even."

We show $P(n)$ for all $n \geq 2$ by induction on n.

Base Case ($\boldsymbol{n = 2}$): <todo>

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $\mathrm{k} \geq 2$.

Inductive Step:

<todo>

We conclude $P(k+1)$. Therefore, $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Setup

Let $P(n)$ be "$g(n)$ is even."

We show $P(n)$ for all $n \geq 2$ by induction on n.

Base Case ($n = 2$): $g(n) = 2$ by definition. 2 is even, so we have $P(2)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $\mathrm{k} \geq 2$.

Inductive Step: We show $P(k+1)$. Consider $g(k+1)$. By definition of $g(\cdot)$, $g(k+1) = g(k)^2 + 3g(k)$. By inductive hypothesis, $g(k)$ is even, so it equals $2j$ for some integer j. Plugging in we have:

$$g(k+1) = (2j)^2 + 3(2j) = 2\left(2j^2\right) + 2(3j) = 2(2j^2 + 3j).$$

Since j is an integer, $2j^2 + 3j$ is also an integer, and $g(k+1)$ is even.

Therefore, $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n.
2. Base Case: Show $P(b)$ i.e. show the base case
3. Inductive Hypothesis: Suppose $P(k)$ for an arbitrary $k \geq b$.
4. Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \to P(k+1)$)
5. Conclude by saying $P(n)$ is true for all $n \geq b$ by the principle of induction.

Let's Try Another Induction Proof

Fundamental Theorem of Arithmetic

Every positive integer greater than 1 has a unique prime factorization.

Uniqueness is hard. Let's just show existence.

I.e.

Claim: Every positive integer greater than 1 can be written as a product of primes.

Induction on Primes.

Let $P(n)$ be "n can be written as a product of primes."

We show $P(n)$ for all $n \geq 2$ by induction on n.

Base Case ($\boldsymbol{n = 2}$): 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary integer $k \geq 2$.

Inductive Step:
Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

Case 2, $k + 1$ is composite:

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

We’re Stuck

We can divide $k+1$ up into smaller pieces (say s, t such that $st = k+1$ with $2 \leq s < k+1$ and $2 \leq t < k+1$

Is $P(s)$ true? Is $P(t)$ true?

I mean…it would be…

But in the inductive step we don’t have it…

Let’s add it to our inductive hypothesis.

Induction on Primes

Let $P(i)$ be "i can be written as a product of primes."

We show $P(n)$ for all $n \geq 2$ by induction on n.

Base Case ($\boldsymbol{n = 2}$): 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

Inductive Hypothesis:

Inductive Step:
Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

Case 2, $k + 1$ is composite:

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

Induction on Primes

Let $P(i)$ be "i can be written as a product of primes."

We show $P(n)$ for all $n \geq 2$ by induction on n.

Base Case ($\boldsymbol{n = 2}$): 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

Inductive Hypothesis: Suppose $P(2), \ldots, P(k)$ hold for an arbitrary integer $k \geq 2$.

Inductive Step:
Case 1, $k+1$ is prime: then $k+1$ is automatically written as a product of primes.

Case 2, $k+1$ is composite: We can write $k+1 = st$ for s, t nontrivial divisors (i.e. $2 \leq s < k+1$ and $2 \leq t < k+1$). By inductive hypothesis, we can write s as a product of primes $p_1 \cdot \ldots p_j$ and t as a product of primes $q_1 \cdots q_\ell$. Multiplying these representations, $k+1 = p_1 \cdots p_j \cdot q_1 \cdots q_\ell$, which is a product of primes.

Therefore $P(k+1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

Strong Induction

That hypothesis where we assume $P(\text{base case}), \ldots, P(k)$ instead of just $P(k)$ is called a strong inductive hypothesis.

Strong induction is the same fundamental idea as weak ("regular") induction.

$P(0)$ is true.
And $P(0) \to P(1)$, so $P(1)$.
And $P(1) \to P(2)$, so $P(2)$.
And $P(2) \to P(3)$, so $P(3)$.
And $P(3) \to P(4)$, so $P(4)$.
…

$P(0)$ is true.
And $P(0) \to P(1)$, so $P(1)$.
And $[\mathrm{P}(0) \wedge P(1)] \to P(2)$, so $P(2)$.
And $[\mathrm{P}(0) \wedge \cdots \wedge P(2)] \to P(3)$, so $P(3)$.
And$[\mathrm{P}(0) \wedge \cdots \wedge P(3)] \to P(4)$, so $P(4)$.
…

Making Induction Proofs Pretty

All of our **strong** induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n.
2. Base Case: Show $P(b)$ i.e. show the base case
3. Inductive Hypothesis: Suppose $\mathrm{P(b)} \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq b$.
4. Inductive Step: Show $P(k+1)$ (i.e. get $[\mathrm{P(b)} \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$)
5. Conclude by saying $P(n)$ is true for all $n \geq b$ by the principle of induction.

Strong Induction vs. Weak Induction

Think of strong induction as “my recursive call might be on LOTS of smaller values” (like mergesort – you cut your array in half)

Think of weak induction as “my recursive call is always on one step smaller.”

Practical advice:

A strong hypothesis isn’t wrong when you only need a weak one (but a weak one is wrong when you need a strong one). Some people just always write strong hypotheses. But it’s easier to typo a strong hypothesis.

Robbie leaves a blank spot where the IH is, and fills it in after the step.

Practical Advice

How many base cases do you need?

Always at least one.

If you're analyzing recursive code or a recursive function, at least one for each base case of the code/function.

If you always go back s steps, at least s consecutive base cases.

Enough to make sure every case is handled.

Let’s Try Another! Stamp Collecting

I have 4 cent stamps and 5 cent stamps (as many as I want of each). Prove that I can make exactly n cents worth of stamps for all $n \geq 12$.

Try for a few values.

Then think…how would the inductive step go?

Stamp Collection (attempt)

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n.

Base Case:

12 cents can be made with three 4 cent stamps.

Inductive Hypothesis Suppose [maybe some other stuff and] $P(k)$, for an arbitrary $k \geq 12$.

Inductive Step:

We want to make $k + 1$ cents of stamps. By IH we can make $k - 3$ cents exactly with stamps. Adding another 4 cent stamp gives exactly $k + 1$ cents.

Stamp Collection

Is the proof right?

How do we know $P(13)$

We're not the base case, so our inductive hypothesis assumes $P(12)$, and then we say if $P(9)$ then $P(13)$.

Wait a second….

If you go back s steps every time, you need s base cases.

Or else the first few values aren't proven.

Stamp Collection

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n.

Base Case:

12 cents can be made with three 4 cent stamps.
13 cents can be made with two 4 cent stamps and one 5 cent stamp.
14 cents can be made with one 4 cent stamp and two 5 cent stamps.
15 cents can be made with three 5 cent stamps.

Inductive Hypothesis Suppose $\mathrm{P}(12) \wedge P(13) \wedge \cdots \wedge P(k)$, for an arbitrary $k \geq 15$.

Inductive Step:

We want to make $k+1$ cents of stamps. By IH we can make $k-3$ cents exactly with stamps. Adding another 4 cent stamp gives exactly $k+1$ cents.

A good last check

After you've finished writing an inductive proof, pause.

If your inductive step always goes back s steps, you need s base cases (otherwise $b + 1$ will go back before the base cases you've shown). And make sure your inductive hypothesis is strong enough.

If your inductive step is going back a varying (unknown) number of steps, check the first few values above the base case, make sure your cases are really covered. And make sure your IH is strong.

Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n.

2. Base Cases: Show $P(b_{min}), P(b_{min+1}) \dots P(b_{max})$ i.e. show the base cases

3. Inductive Hypothesis: Suppose $P(b_{min}) \wedge P(b_{min} + 1) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq b_{max}$. (The smallest value of k assumes **all** bases cases, but nothing else)

4. Inductive Step: Show $P(k+1)$ (i.e. get $[\mathrm{P}(\mathrm{b}_{min}) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$)

5. Conclude by saying $P(n)$ is true for all $n \geq b_{min}$ by the principle of induction.

Stamp Collection, Done Wrong

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n.

Base Case:

12 cents can be made with three 4 cent stamps.
Inductive Hypothesis Suppose $P(k)$, $k \geq 12$.

Inductive Step:

We want to make $k + 1$ cents of stamps. By IH we can make k cents exactly with stamps. Replace one of the 4 cent stamps with a 5 cent stamp.

$P(n)$ holds for all n by the principle of induction.

Stamp Collection, Done Wrong

What if the starting point doesn't have any 4 cent stamps?

Like, say, 15 cents = 5+5+5.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

[Define $P(n)$]

Base Case

Inductive Hypothesis

Inductive Step

[conclusion]

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step:

Target: $P(k + 1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3|(2^{2k}-1)$. i.e. there is an integer j such that $3j = 2^{2k} - 1$.

$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1$

FORCE the expression in your IH to appear

Target: $P(k+1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3|(2^{2k}-1)$. i.e. there is an integer j such that $3j = 2^{2k} - 1$.

$2^{2(k+1)} - 1 = 4(2^{2k} - 1 + 1) - 1 = 4\left(2^{2k} - 1\right) + 4 - 1$

By IH, we can replace $2^{2k} - 1$ with $3j$ for an integer j

$2^{2(k+1)} - 1 = 4(3j) + 4 - 1 = 3(4j) + 3 = 3(4j + 1)$

Since $4j + 1$ is an integer, we meet the definition of divides and we have:

Target: $P(k + 1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

That inductive step might still seem like magic.

It sometimes helps to run through examples, and look for patterns:

$2^{2\cdot 0} - 1 = 0 = 3 \cdot 0$

$2^{2\cdot 1} - 1 = 3 = 3 \cdot 1$

$2^{2\cdot 2} - 1 = 15 = 3 \cdot 5$

$2^{2\cdot 3} - 1 = 63 = 3 \cdot 21$

$2^{2\cdot 4} - 1 = 255 = 3 \cdot 85$

$2^{2\cdot 5} - 1 = 1023 = 3 \cdot 341$

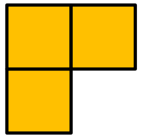
The divisor goes from k to $4k+1$

$$0 \to 4 \cdot 0 + 1 = 1$$
$$1 \to 4 \cdot 1 + 1 = 5$$
$$5 \to 4 \cdot 5 + 1 = 21$$

…

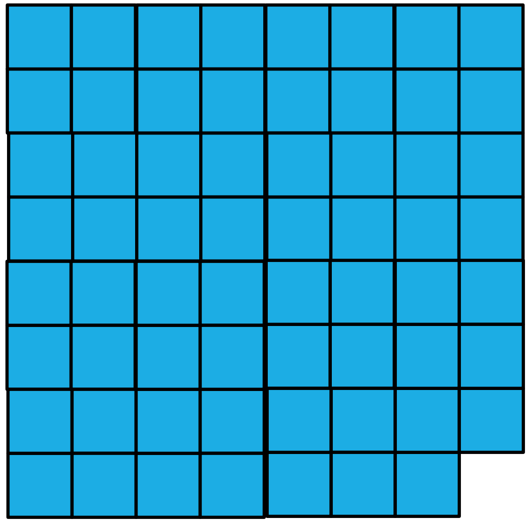
That might give us a hint that $4k+1$ will be in the algebra somewhere, and give us another intermediate target.

Even more practice



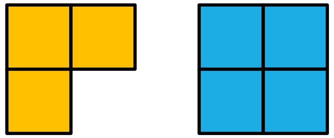
I've got a bunch of these 3 piece tiles.

I want to fill a 2^nx2^n grid ($n \geq 1$) with the pieces, except for a 1x1 spot in a corner.



Gridding (not a full proof, just intuition)

Base Case: $n = 1$



Inductive hypothesis: Suppose you can tile a 2^kx2^k grid, except for a corner.

Inductive step: 2^{k+1}x2^{k+1}, divide into quarters. By IH can tile…