Functions on Strings

Since strings are defined recursively, most functions on strings are as well.

Length:

len(ε)=0;

len(wa)=len(w)+1 for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$;
$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation

$x \cdot \varepsilon = x$ for all $x \in \Sigma^*$;
$x \cdot (wa) = (x \cdot w)a$ for $w \in \Sigma^*, a \in \Sigma$

Number of c's in a string

$\#_c(\varepsilon) = 0$
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$;
$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*, a \in \Sigma \setminus \{c\}$.

Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

Define Let $P(y)$ be "for all $x \in \Sigma^*$ len(x·y)=len(x) + len(y)."

We prove $P(y)$ for all $y \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

Let y be an arbitrary string not covered by the base case. By the exclusion rule, $y = wa$ for a string w and character a.

Inductive Hypothesis: Suppose $P(w)$

Inductive Step: Let x be an arbitrary string.

len(xy)=len(xwa) =len(xw)+1 (by definition of len)

=len(x) + len(w) + 1 (by IH)

=len(x) + len(wa) (by definition of len)

Therefore, len(xy)=len(x) + len(y), as required.

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of y, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

Σ^*:Basis: $\varepsilon \in \Sigma^*$.

Recursive: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$

Structural Induction Template

1. Define $P()$ Show that $P(x)$ holds for all $x \in S$. State your proof is by structural induction.

2. Base Case: Show $P(x)$
[Do that for every base cases x in S.]

Let y be an arbitrary element of S not covered by the base cases. By the exclusion rule, $y =$<recursive rules>

3. Inductive Hypothesis: Suppose $P(x)$
[Do that for every x listed as in S in the recursive rules.]

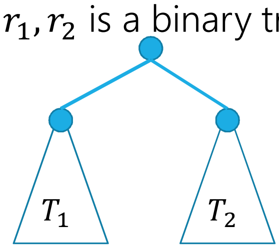
4. Inductive Step: Show $P()$ holds for y.
[You will need a separate case/step for every recursive rule.]

5. Therefore $P(x)$ holds for all $x \in S$ by the principle of induction.

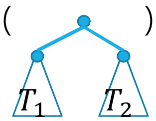
Binary Trees

Basis: A single node is a rooted binary tree.

Recursive Step: If T_1 and T_2 are rooted binary trees with roots r_1 and r_2, then a tree rooted at a new node, with children r_1, r_2 is a binary tree.

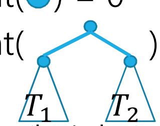


size()=1

size() =

size(T_1) + size(T_2) + 1

height() = 0

height() =

1+max(height(T_1),height(T_2))