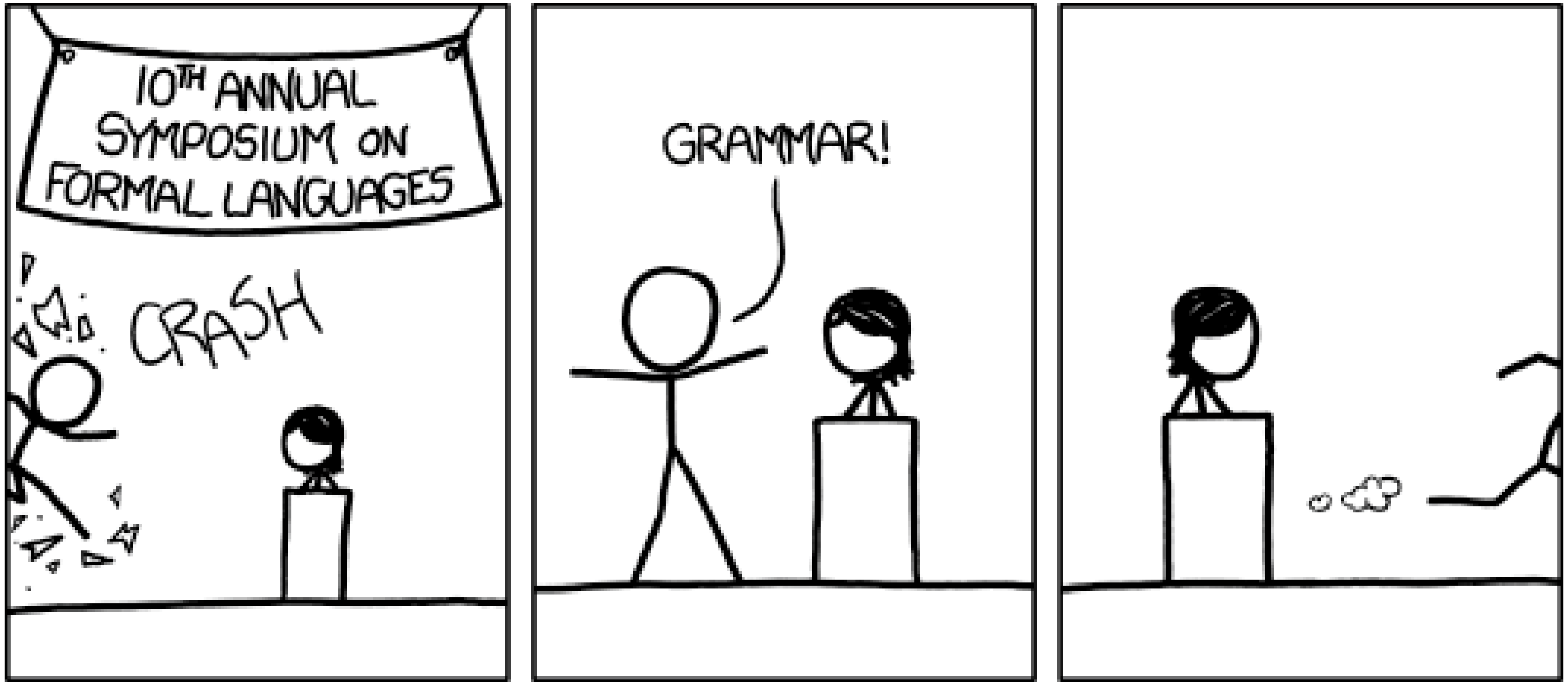


[Audience looks around] “What just happened?” “There must be some context we’re missing.”
xkcd.com/1090

CFGs
Relations

CSE 311 Autumn 2023
Lecture 22

Announcements

Don't forget about HW6 (due Wednesday, but remember late days are different; see the homework pdf).

Today, CSE 403 6PM Jacob will have "how to get started" OH.

CC22 out tonight

Slides for Wednesday, CC23 will be available tomorrow morning; if you're traveling early, you can work ahead.

OH are changed this week (none Thurs/Fri; some Wed switched to zoom)

HW7 will be available Wednesday if you want to work ahead.

Context Free Grammars

Examples

$S \rightarrow 0S0|1S1|0|1|\varepsilon$

The set of all binary palindromes

$S \rightarrow 0S|S1|\varepsilon$

The set of all strings with any 0's coming before any 1's (i.e. 0^*1^*)

$S \rightarrow (S)|SS|\varepsilon$

Balanced parentheses

$S \rightarrow AB$

$A \rightarrow 0A1|\varepsilon$

$B \rightarrow 1B0|\varepsilon$ $\quad \{0^j1^{j+k}0^k : j,k \geq 0\}$

Multiple ways of generating strings

Generate $2 + 3 * 4$in two different ways

$E \Rightarrow E + E \Rightarrow E + E * E \Rightarrow 2 + E * E \Rightarrow 2 + 3 * E \Rightarrow 2 + 3 * 4$

$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow 2 + E * E \Rightarrow 2 + 3 * E \Rightarrow 2 + 3 * 4$

What did we mean by these being different? They represent different meanings mathematically.

One says “you’re adding together two numbers: 2 and (whatever 3*4 is)”

The other says “you’re multiplying two numbers: (whatever 2+3 is) and 4”

Those have different meanings!

Parse Trees—remember where parentheses go

Suppose a context free grammar G generates a string x

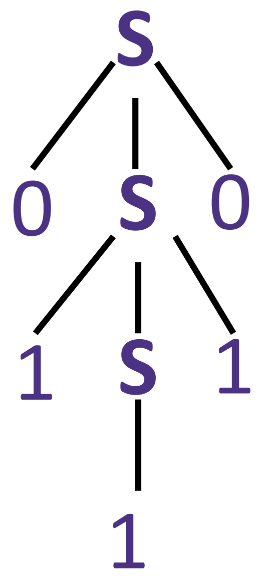
A parse tree of x for G has

Rooted at S (start symbol)

Children of every A node are labeled with the characters of w for some $A \rightarrow w$

Reading the leaves from left to right gives x.

$S \rightarrow 0S0|1S1|0|1|\varepsilon$

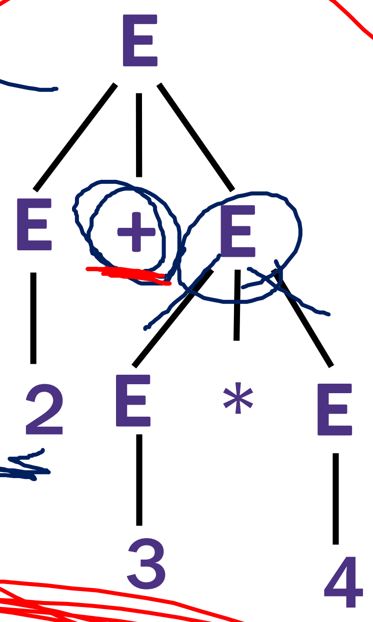


Back to the arithmetic

$E \rightarrow E + E | E * E | (E) | x | y | z | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9$

Two parse trees for $2 + 3 * 4$

14



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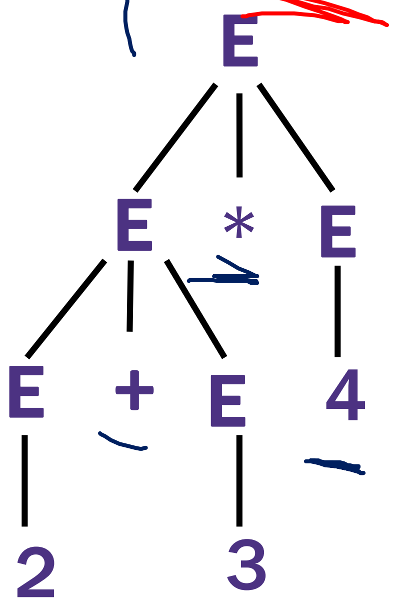


$E \Rightarrow E + E$
$\Rightarrow E + E * E$
$\Rightarrow E + 3 * E$
$\Rightarrow 2 + 3 * E$
$\Rightarrow 2 + 3 * 4$

"parsing"

Why do we care about parsing?

$2 + 3 * 4$ can only mean one thing!

If I write these symbols in a program, we need to make sure we know which one to do.

The first grammar we saw was “ambiguous” it allows the same string to “mean” two different things.

Sometimes you can fix that!

How do we encode order of operations

If we want to keep "in order" we want there to be only one possible parse tree.

Differentiate between "things to add" and "things to multiply"

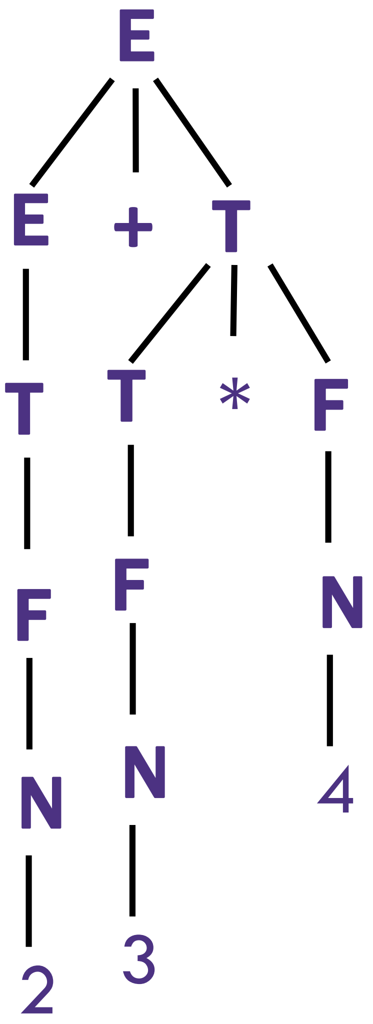
Only introduce a * sign after you've eliminated the possibility of introducing another + sign in that area.

$E \rightarrow T|E+T$

$T \rightarrow F|T*F$

$F \rightarrow (E)|N$

$N \rightarrow x|y|z|0|1|2|3|4|5|6|7|8|9$



How do Computer Scientists use CFGs?

Most programming languages define valid programs as “strings that fit a CFG”
That makes sure Java breaks down math expressions correctly! And also code like this:

"dangling else"

```
if(i>0)
   if(j>0)
      System.out.println("hi");
   else(k>1)
      System.out.println("bye");
```

```
if(i>0)
   if(j>0)
      System.out.println("hi");
else(k>1)
   System.out.println("bye");
```

The else could be attached to either “if”! Java needs a rule to decide which it goes with.
Java’s convention makes the one on the left the intuitive whitespace.
(You as a programmer should put braces so the humans reading your code don’t have to wonder!)

CFGs in practice

Used to define programming languages.

Often written in Backus-Naur Form – just different notation

Variables are <names-in-brackets> (or sometimes without)

like <if-then-else-statement>, <condition>, <identifier>

$\rightarrow$ is replaced with ::= or :

BNF for C (no <...> and uses : instead of ::=)

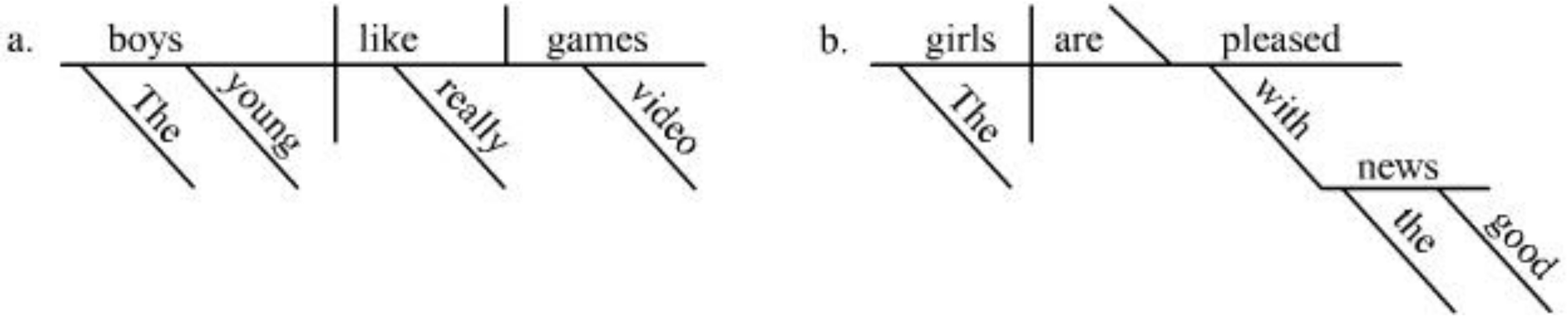
```
statement:
  ((identifier | "case" constant-expression | "default") ":")*
  (expression? ";" |
   block |
   "if" "(" expression ")" statement |
   "if" "(" expression ")" statement "else" statement |
   "switch" "(" expression ")" statement |
   "while" "(" expression ")" statement |
   "do" statement "while" "(" expression ")" ";" |
   "for" "(" expression? ";" expression? ";" expression? ")" statement |
   "goto" identifier ";" |
   "continue" ";" |
   "break" ";" |
   "return" expression? ";"
  )

block: "{" declaration* statement* "}"

expression:
  assignment-expression%

assignment-expression: (
    unary-expression (
      "=" | "*=" | "/=" | "%=" | "+=" | "-=" | "<<=" | ">>=" | "&=" |
      "^=" | "|="
    )
  )* conditional-expression

conditional-expression:
  logical-OR-expression ( "?" expression ":" conditional-expression )?
```

Some fun CFG applications

If we have time

Parse Trees

Remember diagramming sentences in middle school?



<sentence>::=<noun phrase><verb phrase>

<noun phrase>::=<determiner><adjective><noun>

<verb phrase>::=<verb><adverb>|<verb><object>

<object>::=<noun phrase>

Parse Trees

<sentence>::=<noun phrase><verb phrase>

<noun phrase>::=<determiner><adjective><noun>

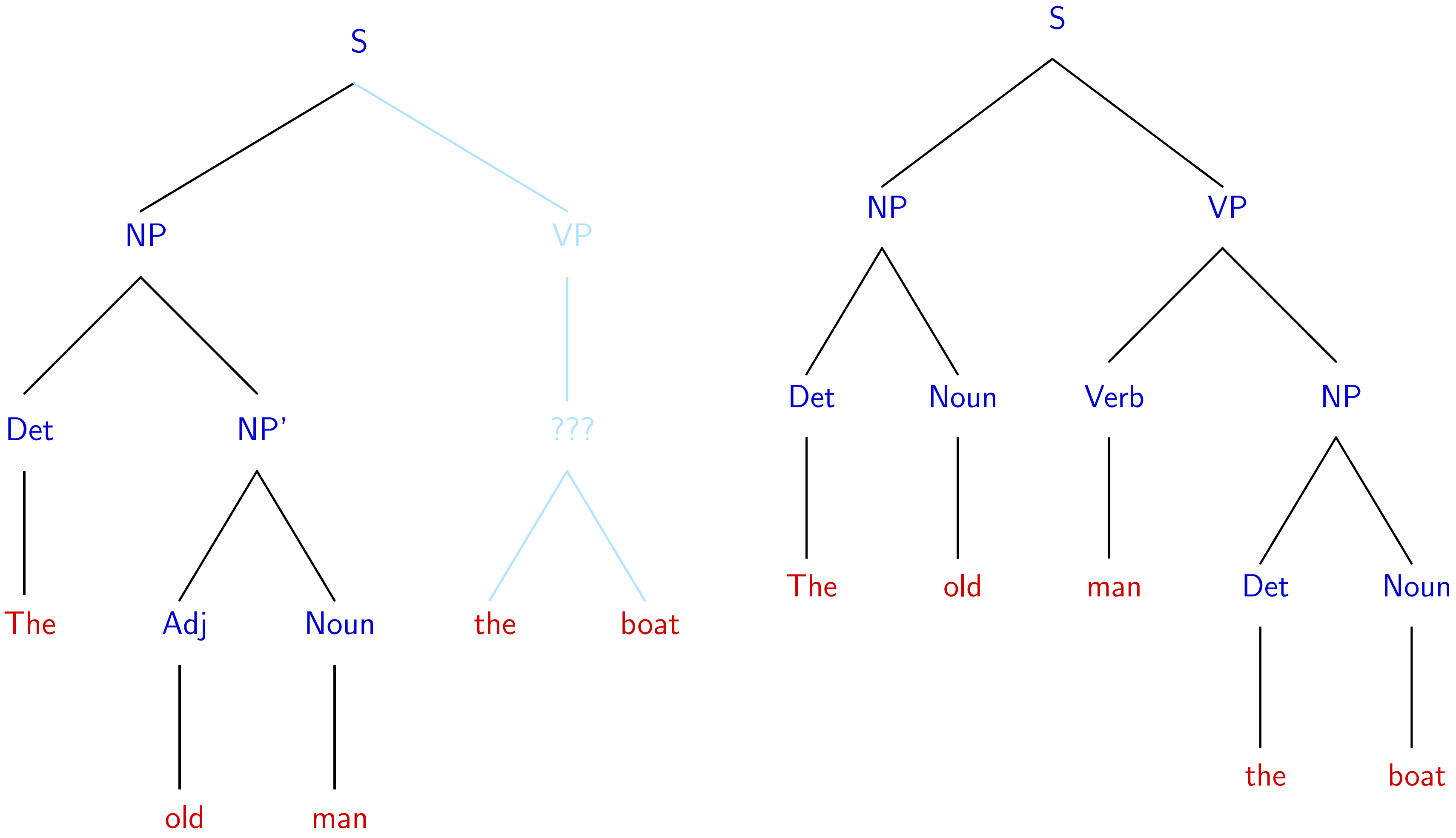
<verb phrase>::=<verb><adverb>|<verb><object>

<object>::=<noun phrase>

The old man the boat.

"garden path sentence"

The old man the boat



English is ambiguous

(Most of 'standard') English can be represented as a context free grammar.

It's not perfect (ask Robbie details later).

The grammar is ambiguous! That is, there are sentences which have multiple valid parsings (multiple meanings).

Can you find multiple meanings of this sentence:
"Place these 3 exercise balls on the mat at the top of the hill."

[See this video]

The Important Takeaways

Power of Context Free Languages

There are languages CFGs can express that regular expressions can't

e.g. palindromes

What about vice versa – is there a language that a regular expression can represent that a CFG can't?

No!

Are there languages even CFGs cannot represent?

Yes!

$\{0^k 1^j 2^k 3^j | j, k \geq 0\}$ cannot be written with a context free grammar.

Takeaways

CFGs and regular expressions gave us ways of succinctly representing sets of strings

- Regular expressions super useful for representing things you need to search for
- CFGs represent complicated languages like "java code with valid syntax"

This week, two more tools for our toolbox (relations, graphs)

After Thanksgiving, (mathematical representations of) Tiny computers! And how they relate to regular expressions and CFGs.

Relations and Graphs

Relations $\{1,2\} \times \{3,4\} = \{(1,3),(1,4),(2,3),(2,4)\}$

Relations

A (binary) relation from A to B is a subset of $A \times B$
A (binary) relation on A is a subset of $A \times A$

$(0,5)$ $\cancel{(5,0)}$

Wait what?

$\leq$ is a relation on $\mathbb{Z}$.

"$3 \leq 4$" is a way of saying "3 relates to 4" (for the $\leq$ relation)

$(3,4)$ is an element of the set that defines the relation.

Relations, Examples

It turns out, they've been here the whole time

$<$ on $\mathbb{R}$ is a relation

I.e. $\{(x, y) : x < y \text{ and } x, y \in \mathbb{R}\}$.

$=$ on Σ^* is a relation

i.e. $\{(x, y) : x = y \text{ and } x, y \in \Sigma^*\}$

For your favorite function f, you can define a relation from its domain to its co-domain

i.e. $\{(x, y) : f(x) = y\}$

"x when squared gives y" is a relation

i.e. $\{(x, y): x^2 = y, x, y \in \mathbb{R}\}$

Relations, Examples

Fix a universal set $\mathcal{U}$.

$\subseteq$ is a relation. What's it on?

$\mathcal{P}(\mathcal{U})$
The set of all subsets of $\mathcal{U}$

More Relations

$R_1 = \{(a,1),(a,2),(b,1),(b,3),(c,3)\}$

Is a relation (you can define one just by listing what relates to what)

Equivalence mod 5 is a relation.

$\{(x,y) : x \equiv y (mod\ 5)\}$

We'll also say "x relates to y if and only if they're congruent mod 5"

Properties of relations

$3 = 2 + 1$

$2 + 1 = 3$

What do we do with relations? Usually we prove properties about them.

$2 \leq 3 \quad 3 \leq 5 \longrightarrow 2 \leq 5$

Symmetry
A binary relation R on a set S is "symmetric" iff
for all $a, b \in S$, $[(a, b) \in R \to (b, a) \in R]$

$=$ on Σ^* is symmetric, for all $a, b \in \Sigma^*$ if $a = b$ then $b = a$.
$\subseteq$ is not symmetric on $\mathcal{P}(\mathcal{U})$ – $\{1,2,3\} \subseteq \{1,2,3,4\}$ but $\{1,2,3,4\} \not\subseteq \{1,2,3\}$

Transitivity
A binary relation R on a set S is "transitive" iff
for all $a, b, c \in S$, $[(a, b) \in R \wedge (b, c) \in R \to (a, c) \in R]$

$=$ on Σ^* is transitive, for all $a, b, c \in \Sigma^*$ if $a = b$ and $b = c$ then $a = c$.
$\subseteq$ is transitive on $\mathcal{P}(\mathcal{U})$ – for any sets A, B, C if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
$\in$ is not a transitive relation – $1 \in \{1,2,3\}$, $\{1,2,3\} \in \mathcal{P}(\{1,2,3\})$ but $1 \notin \mathcal{P}(\{1,2,3\})$.

$\{1\} \subseteq \{1,2\}, \quad \{1,2\} \subseteq \{1,2,3\}$

Warm up

Show that $a \equiv b\ (mod\ n)$ if and only if $b \equiv a (mod\ n)$

$a \equiv b(mod\ n) \leftrightarrow n|(b-a) \leftrightarrow nk = b - a(\text{for } k \in \mathbb{Z}) \leftrightarrow$

$n(-k) = a - b(\text{for } -\mathrm{k} \in \mathbb{Z}) \leftrightarrow n|(a-b) \leftrightarrow b \equiv a(mod\ n)$

This was a proof that the relation $\{(a, b) : a \equiv b(mod\ n)\}$ is symmetric!

It was actually overkill to show if and only if. Showing just one direction turns out to be enough!

this is the form of the division theorem for $(a-n)\%n$. Since the division theorem guarantees a unique integer, $(a-n)\%n = (a\%n)$

What about transitivity?

You did this as a homework problem!

Divides is a transitive relation!

If $p|q$ and $q|r$ then $p|r$.

More Properties of relations

$3 \le 5$

$5 \not\le 3$

What do we do with relations? Usually we prove properties about them.

$2 \le 2$ $\quad$ $2 \le 2$

Antisymmetry

A binary relation R on a set S is "antisymmetric" iff for all $a, b \in S$, $[(a,b) \in R \wedge a \neq b \rightarrow (\mathbf{b},\mathbf{a}) \notin R]$

$\le$ is antisymmetric on $\mathbb{Z}$

Reflexivity

A binary relation R on a set S is "reflexive" iff for all $a \in S$, $[(a,a) \in R]$

$\le$

$3 \le 3$

$\le$ is reflexive on $\mathbb{Z}$

You've proven antisymmetry too!

(a) Prove that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

Solution:

Suppose that $a \mid b$ and $b \mid a$, where a, b are integers. By the definition of divides, we have $a \neq 0$, $b \neq 0$ and $b = ka, a = jb$ for some integers k, j. Combining these equations, we see that $a = j(ka)$.

Then, dividing both sides by a, we get $1 = jk$. So, $\frac{1}{j} = k$. Note that j and k are integers, which is only possible if $j, k \in \{1, -1\}$. It follows that $b = -a$ or $b = a$.

Antisymmetry

A binary relation R on a set S is "antisymmetric" iff for all $a, b \in S$, $[(a, b) \in R \wedge a \neq b \rightarrow (\mathrm{b}, \mathrm{a}) \notin R]$

You showed | is antisymmetric on $\mathbb{Z}^{+}$ in section 5.

for all $a, b \in S$, $[(a, b) \in R \wedge (\mathrm{b}, \mathrm{a}) \in R \rightarrow a = b]$ is equivalent to the definition in the box above

The box version is easier to understand, the other version is usually easier to prove.

Try a few of your own

Decide whether each of these relations are

Reflexive, symmetric, antisymmetric, and transitive.

$\subseteq$ on $\mathcal{P}(\mathcal{U})$

$\geq$ on $\mathbb{Z}$

$>$ on $\mathbb{R}$

$|$ on $\mathbb{Z}^+$

$|$ on $\mathbb{Z}$

$\equiv (mod\ 3)$ on $\mathbb{Z}$

Symmetry: for all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{S}$, $[(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R} \rightarrow (\mathbf{b}, \mathbf{a}) \in \boldsymbol{R}]$

Antisymmetry: for all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{S}$, $[(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R} \wedge \boldsymbol{a} \neq \boldsymbol{b} \rightarrow (\mathbf{b}, \mathbf{a}) \notin \boldsymbol{R}]$

Transitivity: for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{S}$, $[(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R} \wedge (\boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{R} \rightarrow (\mathbf{a}, \mathbf{c}) \in \boldsymbol{R}]$

Reflexivity: for all $\boldsymbol{a} \in \boldsymbol{S}$, $[(\boldsymbol{a}, \boldsymbol{a}) \in \boldsymbol{R}]$

Try a few of your own

Symmetry: for all $a, b \in S$, $[(a, b) \in R \to (\mathrm{b}, \mathrm{a}) \in R]$

Antisymmetry: for all $a, b \in S$, $[(a, b) \in R \wedge a \neq b \to (\mathrm{b}, \mathrm{a}) \notin R]$

Transitivity: for all $a, b, c \in S$, $[(a, b) \in R \wedge (b, c) \in R \to (\mathrm{a}, \mathrm{c}) \in R]$

Reflexivity: for all $a \in S$, $[(a, a) \in R]$

Decide whether each of these relations are

Reflexive, symmetric, antisymmetric, and transitive.

$\subseteq$ on $\mathcal{P}(\mathcal{U})$ reflexive, antisymmetric, transitive

$\geq$ on $\mathbb{Z}$ reflexive, antisymmetric, transitive

$>$ on $\mathbb{R}$ antisymmetric, transitive

$|$ on $\mathbb{Z}^+$ reflexive, antisymmetric, transitive

$|$ on $\mathbb{Z}$ reflexive, transitive

$\equiv (mod\ 3)$ on $\mathbb{Z}$ reflexive, symmetric, transitive