CSE 311 Section 4

English Proofs & Set Theory

Administrivia

Announcements & Reminders

- HW2
  - If you think something was graded incorrectly, submit a regrade request!
- HW3 due tomorrow 10/21 @ 10PM on Gradescope
  - Use late days if you need them!
- HW4
  - Due Friday 10/28 @ 10pm

References

- Helpful reference sheets can be found on the course website!
  - https://courses.cs.washington.edu/courses/cse311/23wi/resources/
- How to LaTeX (found on Assignments page of website):
  - https://courses.cs.washington.edu/courses/cse311/23wi/assignments/HowToLaTeX.pdf
- Set Reference Sheet
  - https://courses.cs.washington.edu/courses/cse311/23wi/resources/reference-sets.pdf
- Number Theory Reference Sheet
  - https://courses.cs.washington.edu/courses/cse311/23wi/resources/reference-number-theory.pdf
- Plus more!

English Proofs

Writing a Proof (symbolically or in English)

- Don’t just jump right in!
- Look at the **claim**, and make sure you know:
  - What every word in the claim means
  - What the claim as a whole means
- Translate the claim in predicate logic.
- Next, write down the **Proof Skeleton**:
  - Where to start
  - What your target is
- Then once you know what claim you are proving and your starting point and ending point, you can finally write the proof!

Helpful Tips for English Proofs

- Start by introducing your assumptions
  - Introduce variables with "let"
    - "Let x be an arbitrary prime number…"
  - Introduce assumptions with "suppose"
    - "Suppose that $y \in A \land y \notin B$…"
- When you supply a value for an existence proof, use "Consider"
  - "Consider $x = 2$…"
- **ALWAYS** state what type your variable is (integer, set, etc.)
- Universal Quantifier means variable must be arbitrary
- Existential Quantifier means variable can be specific

Problem 2 – Just the Setup

For each of these statements,

- Translate the sentence into predicate logic.
- Write the first few sentences and last few sentences of the English proof.

a) The product of an even integer and an odd integer is even.

b) There is an integer x such that $x^2 > 10$ and $3x$ is even.

c) For every integer n, there is a prime number p greater than n.

d) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ for any sets A, B, C.

Work on parts (b) and (c) with the people around you, and then we'll go over it together!

Problem 2 – Just the Setup

b) There is an integer x such that $x^2 > 10$ and $3x$ is even.

Problem 2 – Just the Setup

b) There is an integer x such that $x^2 > 10$ and $3x$ is even.

$\exists x[\text{GreaterThan10}(x^2) \wedge \text{Even}(3x)]$

Problem 2 – Just the Setup

b) There is an integer x such that $x^2 > 10$ and $3x$ is even.

$\exists x[\text{GreaterThan10}(x^2) \wedge \text{Even}(3x)]$

Consider $x = 6$.

….

Problem 2 – Just the Setup

b) There is an integer x such that $x^2 > 10$ and $3x$ is even.

$\exists x[\text{GreaterThan10}(x^2) \wedge \text{Even}(3x)]$

Consider $x = 6$.

. . . .

Then there exists some integer k such that $3 \cdot 6 = 2k$.

Problem 2 – Just the Setup

b) There is an integer x such that $x^2 > 10$ and $3x$ is even.

$\exists x[\text{GreaterThan10}(x^2) \wedge \text{Even}(3x)]$

Consider $x = 6$.

. . . .

Then there exists some integer k such that $3 \cdot 6 = 2k$.

So $6^2 > 10$ and $3 \cdot 6$ is even.

Hence, 6 is the desired x.

Problem 2 – Just the Setup

c) For every integer n, there is a prime number p greater than n.

Problem 2 – Just the Setup

c) For every integer n, there is a prime number p greater than n.

$\forall x \exists y[\mathrm{Prime}(y) \wedge \mathrm{GreaterThan}(y, x)]$

Problem 2 – Just the Setup

c) For every integer n, there is a prime number p greater than n.

$\forall x \exists y[\mathrm{Prime}(y) \wedge \mathrm{GreaterThan}(y, x)]$

Let x be an arbitrary integer.

Problem 2 – Just the Setup

c) For every integer n, there is a prime number p greater than n.

$\forall x \exists y[\text{Prime}(y) \wedge \text{GreaterThan}(y, x)]$

Let x be an arbitrary integer.
Consider $y = p$ (this p is a specific prime)
. . . .

Problem 2 – Just the Setup

c) For every integer n, there is a prime number p greater than n.

$\forall x \exists y[\text{Prime}(y) \wedge \text{GreaterThan}(y, x)]$

Let x be an arbitrary integer.
Consider $y = p$ (this p is a specific prime)
. . . .
So p is prime and $p > x$.
Since x was arbitrary, we have that every integer has a prime number that is greater than it.

Sets

Sets

- A set is an **unordered** group of **distinct** elements
  - Set variable names are capital letters, with lower-case letters for elements

- Set Notation:
  - $a \in A$: "a is in A" or "a is an element of A"
  - $A \subseteq B$: "A is a subset of B", every element of A is also in B
  - $\emptyset$: "empty set", a unique set containing no elements
  - $\mathcal{P}(A)$: "power set of A", the set of all subsets of A including the empty set and A itself

Set Operators

- Subset: $A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
- Equality: $A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \land B \subseteq A$
- Union: $A \cup B = \{x: x \in A \lor x \in B\}$
- Intersection: $A \cap B = \{x: x \in A \land x \in B\}$
- Complement: $\overline{A} = \{x: x \notin A\}$
- Difference: $A \backslash \mathrm{B} = \{x: x \in A \land x \notin B\}$
- Cartesian Product: $A \times B = \{(a, b): a \in A \land b \in B\}$

Problem 3 – How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

a) $A = \{1, 2, 3, 2\}$

b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$

c) $C = A \times (B \cup \{7\})$

d) $D = \emptyset$

e) $E = \{\emptyset\}$

f) $F = \mathcal{P}(\{\emptyset\})$

Work this problem with the people around you, and then we'll go over it together!

Problem 3 – How Many Elements?

a) $A = \{1, 2, 3, 2\}$

b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$

c) $C = A \times (B \cup \{7\})$

d) $D = \emptyset$

e) $E = \{\emptyset\}$

f) $F = \mathcal{P}(\{\emptyset\})$

Problem 3 – How Many Elements?

a) $A = \{1, 2, 3, 2\}$ $3, A = \{1, 2, 3\}$

b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$

c) $C = A \times (B \cup \{7\})$

d) $D = \emptyset$

e) $E = \{\emptyset\}$

f) $F = \mathcal{P}(\{\emptyset\})$

Problem 3 – How Many Elements?

a) $A = \{1, 2, 3, 2\}$ $3, A = \{1, 2, 3\}$

b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$ $2, B = \{\emptyset, \{\emptyset\}\}$

c) $C = A \times (B \cup \{7\})$

d) $D = \emptyset$

e) $E = \{\emptyset\}$

f) $F = \mathcal{P}(\{\emptyset\})$

Problem 3 – How Many Elements?

a) $A = \{1, 2, 3, 2\}$ $3, A = \{1, 2, 3\}$

b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$ $2, B = \{\emptyset, \{\emptyset\}\}$

c) $C = A \times (B \cup \{7\})$ $9, C = \{1, 2, 3\} \times \{7, \emptyset, \{\emptyset\}\}$

d) $D = \emptyset$

e) $E = \{\emptyset\}$

f) $F = \mathcal{P}(\{\emptyset\})$

Problem 3 – How Many Elements?

a) $A = \{1, 2, 3, 2\}$ 3, $A = \{1, 2, 3\}$

b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$ 2, $B = \{\emptyset, \{\emptyset\}\}$

c) $C = A \times (B \cup \{7\})$ 9, $C = \{1, 2, 3\} \times \{7, \emptyset, \{\emptyset\}\}$

d) $D = \emptyset$ 0

e) $E = \{\emptyset\}$

f) $F = \mathcal{P}(\{\emptyset\})$

Problem 3 – How Many Elements?

a) $A = \{1, 2, 3, 2\}$ $3, A = \{1, 2, 3\}$

b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$ $2, B = \{\emptyset, \{\emptyset\}\}$

c) $C = A \times (B \cup \{7\})$ $9, C = \{1, 2, 3\} \times \{7, \emptyset, \{\emptyset\}\}$

d) $D = \emptyset$ 0

e) $E = \{\emptyset\}$ 1

f) $F = \mathcal{P}(\{\emptyset\})$

Problem 3 – How Many Elements?

a) $A = \{1, 2, 3, 2\}$ $3, A = \{1, 2, 3\}$

b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$ $2, B = \{\emptyset, \{\emptyset\}\}$

c) $C = A \times (B \cup \{7\})$ $9, C = \{1, 2, 3\} \times \{7, \emptyset, \{\emptyset\}\}$

d) $D = \emptyset$ 0

e) $E = \{\emptyset\}$ 1

f) $F = \mathcal{P}(\{\emptyset\})$ $2, F = \{\emptyset, \{\emptyset\}\}$

Set Proofs

Subset Proofs

One of the most common types of proofs you will be asked to write involving sets is a subset proof. That is, you will be asked to prove that $A \subseteq B$. We always approach these proofs with the same proof skeleton:

Let x be an arbitrary element of A, so $x \in A$.
$\ldots$ some steps using set definitions to show that x must also be in B$\ldots$
Thus, $x \in B$
Since x was arbitrary, $A \subseteq B$.

Set Equality Proofs

Another common type of set proof is proving that $A = B$. The trick here is that this is secretly just two subset proofs! We need to show both that $A \subseteq B$ and $B \subseteq A$. Again, we will always use the same proof skeleton:

Let x be an arbitrary element of A, so $x \in A$.
$\dots$ Thus, $x \in B$
Since x was arbitrary, $A \subseteq B$.
Let y be an arbitrary element of B, so $y \in B$.
$\dots$ Thus, $y \in A$
Since y was arbitrary, $\mathrm{B} \subseteq A$.
As we have shown both that $A \subseteq B$ and $B \subseteq A$, therefore $A = B$.

Problem 4 – Set = Set

Prove the following set identities. Write both a formal inference proof and an English proof.

a) Let the universal set be $\mathcal{U}$. Prove $A \cap \bar{B} \subseteq A \setminus B$ for any sets A, B.

b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

Work on part (b) with the people around you, and then we'll go over it together!

Problem 4 – Set = Set

b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

Problem 4 – Set = Set

b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

Let x be an arbitrary element of $(A \cap B) \times C$.

…

Since x was an arbitrary element of $(A \cap B) \times C$ we have proved that $(A \cap B) \times C \subseteq A \times (C \cup D)$ as required.

Problem 4 – Set = Set

b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

Let x be an arbitrary element of $(A \cap B) \times C$.
Then, by definition of Cartesian product, x must be of the form (y, z) where $y \in A \cap B$ and $z \in C$.
…
Since x was an arbitrary element of $(A \cap B) \times C$ we have proved that $(A \cap B) \times C \subseteq A \times (C \cup D)$ as required.

Problem 4 – Set = Set

b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

Let x be an arbitrary element of $(A \cap B) \times C$.
Then, by definition of Cartesian product, x must be of the form (y, z) where $y \in A \cap B$ and $z \in C$.
Since $y \in A \cap B$, $y \in A$ and $y \in B$ by definition of $\cap$; in particular, all we care about is that $y \in A$.
…
Since x was an arbitrary element of $(A \cap B) \times C$ we have proved that $(A \cap B) \times C \subseteq A \times (C \cup D)$ as required.

Problem 4 – Set = Set

b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

Let x be an arbitrary element of $(A \cap B) \times C$.
Then, by definition of Cartesian product, x must be of the form (y, z) where $y \in A \cap B$ and $z \in C$.
Since $y \in A \cap B$, $y \in A$ and $y \in B$ by definition of ∩; in particular, all we care about is that $y \in A$.
Since $z \in C$, by definition of ∪, we also have $z \in C \cup D$.
…
Since x was an arbitrary element of $(A \cap B) \times C$ we have proved that $(A \cap B) \times C \subseteq A \times (C \cup D)$ as required.

Problem 4 – Set = Set

b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

Let x be an arbitrary element of $(A \cap B) \times C$.
Then, by definition of Cartesian product, x must be of the form (y, z) where $y \in A \cap B$ and $z \in C$.
Since $y \in A \cap B$, $y \in A$ and $y \in B$ by definition of ∩; in particular, all we care about is that $y \in A$.
Since $z \in C$, by definition of ∪, we also have $z \in C \cup D$.
Therefore since $y \in A$ and $z \in C \cup D$, by definition of Cartesian product we have $x = (y, z) \in A \times (C \cup D)$.
Since x was an arbitrary element of $(A \cap B) \times C$ we have proved that $(A \cap B) \times C \subseteq A \times (C \cup D)$ as required.

Problem 5 – Set Equality

a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

b) Let $\mathcal{U}$ be the universal set. Show that $\overline{\overline{X}} = X$

Work on part (a) with the people around you, and then we'll go over it together!

Problem 5 – Set Equality

a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

Problem 5 – Set Equality

a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

Let x be an arbitrary element of $A \cap (A \cup B)$.

$\ldots$

Since x was arbitrary, $A \cap (A \cup B) \subseteq A$.

Now let y be an arbitrary member of A. Then $y \in A$. So certainly $y \in A$ or $y \in B$.

$\ldots$

Since y was arbitrary, $A \subseteq A \cap (A \cup B)$.

Therefore $A \cap (A \cup B) = A$, by containment in both directions.

Problem 5 – Set Equality

a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

Let x be an arbitrary element of $A \cap (A \cup B)$.
Then by definition of intersection, $x \in A$ and $x \in A \cup B$. So certainly, $x \in A$.
Since x was arbitrary, $A \cap (A \cup B) \subseteq A$.

Now let y be an arbitrary member of A. Then $y \in A$. So certainly $y \in A$ or $y \in B$.
$\ldots$
Since y was arbitrary, $A \subseteq A \cap (A \cup B)$.

Therefore $A \cap (A \cup B) = A$, by containment in both directions.

Problem 5 – Set Equality

a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

Let x be an arbitrary element of $A \cap (A \cup B)$.
Then by definition of intersection, $x \in A$ and $x \in A \cup B$. So certainly, $x \in A$.
Since x was arbitrary, $A \cap (A \cup B) \subseteq A$.

Now let y be an arbitrary member of A. Then $y \in A$. So certainly $y \in A$ or $y \in B$.
Then by definition of union, $y \in A \cup B$.
$\ldots$
Since y was arbitrary, $A \subseteq A \cap (A \cup B)$.

Therefore $A \cap (A \cup B) = A$, by containment in both directions.

Problem 5 – Set Equality

a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

Let x be an arbitrary element of $A \cap (A \cup B)$.
Then by definition of intersection, $x \in A$ and $x \in A \cup B$. So certainly, $x \in A$.
Since x was arbitrary, $A \cap (A \cup B) \subseteq A$.

Now let y be an arbitrary member of A. Then $y \in A$. So certainly $y \in A$ or $y \in B$.
Then by definition of union, $y \in A \cup B$.
Since $y \in A$ and $y \in A \cup B$, then by definition of intersection, $y \in A \cap (A \cup B)$.
Since y was arbitrary, $A \subseteq A \cap (A \cup B)$.

Therefore $A \cap (A \cup B) = A$, by containment in both directions.

Bonus: Inference Proofs

Inference Proofs

- New way of doing proofs:
  - Write down all the facts we know (givens)
  - Combine the things we know to derive new facts
  - Continue until what we want to show is a fact

- **Modus Ponens**
  - $[(p \to q) \land p] \to q \equiv T$
  - If you have an implication and its hypothesis as facts, you can get the conclusion

- **Direct Proof Rule**
  - Assume x and then eventually get y, you can conclude that $x \to y$

Inference Proof Example

Given $((p \rightarrow q) \wedge (q \rightarrow r))$, show that $(p \rightarrow r)$

1.	$((p \rightarrow q) \wedge (q \rightarrow r))$	Given
2.	$p \rightarrow q$	Eliminate ∧: 1
3.	$q \rightarrow r$	Eliminate ∧: 1
	4.1 p	Assumption
	4.2 q	Modus Ponens: 4.1, 2
	4.3 r	Modus Ponens: 4.2, 3
5.	$p \rightarrow r$	Direct Proof Rule

Problem 8 – Formal Proof (Direct Proof Rule)

Show that $\neg t \rightarrow s$ follows from $t \vee q$, $q \rightarrow r$ and $r \rightarrow s$

Work on this problem with the people around you, and then we'll go over it together!

Problem 8 – Formal Proof (Direct Proof Rule)

Show that $\neg t \to s$ follows from $t \vee q$, $q \to r$ and $r \to s$

1.	$t \vee q$	Given
2.	$q \to r$	Given
3.	$r \to s$	Given
?.	$\neg t \to s$	???

Problem 8 – Formal Proof (Direct Proof Rule)

Show that $\neg t \rightarrow s$ follows from $t \vee q$, $q \rightarrow r$ and $r \rightarrow s$

1.	$t \vee q$	Given
2.	$q \rightarrow r$	Given
3.	$r \rightarrow s$	Given
4.1	$\neg t$	Assumption
?.	$\neg t \rightarrow s$	???

Problem 8 – Formal Proof (Direct Proof Rule)

Show that $\neg t \rightarrow s$ follows from $t \vee q$, $q \rightarrow r$ and $r \rightarrow s$

1.	$t \vee q$		Given
2.	$q \rightarrow r$		Given
3.	$r \rightarrow s$		Given
	4.1	$\neg t$	Assumption
	4.2	q	Eliminate ∨: 1, 4.1
?.	$\neg t \rightarrow s$		???

Problem 8 – Formal Proof (Direct Proof Rule)

Show that $\neg t \rightarrow s$ follows from $t \vee q$, $q \rightarrow r$ and $r \rightarrow s$

1.	$t \vee q$		Given
2.	$q \rightarrow r$		Given
3.	$r \rightarrow s$		Given
	4.1	$\neg t$	Assumption
	4.2	q	Eliminate ∨: 1, 4.1
	4.3	r	Modus Ponens: 4.2, 2
?.	$\neg t \rightarrow s$		???

Problem 8 – Formal Proof (Direct Proof Rule)

Show that $\neg t \rightarrow s$ follows from $t \vee q$, $q \rightarrow r$ and $r \rightarrow s$

1.	$t \vee q$		Given
2.	$q \rightarrow r$		Given
3.	$r \rightarrow s$		Given
	4.1	$\neg t$	Assumption
	4.2	q	Eliminate ∨: 1, 4.1
	4.3	r	Modus Ponens: 4.2, 2
	4.4	s	Modus Ponens: 4.3, 3
?.	$\neg t \rightarrow s$		???

Problem 8 – Formal Proof (Direct Proof Rule)

Show that $\neg t \rightarrow s$ follows from $t \lor q$, $q \rightarrow r$ and $r \rightarrow s$

1.	$t \lor q$		Given
2.	$q \rightarrow r$		Given
3.	$r \rightarrow s$		Given
	4.1	$\neg t$	Assumption
	4.2	q	Eliminate ∨: 1, 4.1
	4.3	r	Modus Ponens: 4.2, 2
	4.4	s	Modus Ponens: 4.3, 3
5.	$\neg t \rightarrow s$		Direct Proof Rule

That's All, Folks!

Thanks for coming to section this week!
Any questions?