Section 04: Formal Proofs and English Translation

1. Formal Spoofs

For each of the following proofs, determine why the proof is incorrect. Then, consider whether the conclusion of the proof is true or not. If it is true, state how the proof could be fixed. If it is false, give a counterexample.

(a) Show that $\exists z\ \forall x\ P(x, z)$ follows from $\forall x\ \exists y\ P(x, y)$.

1.	$\forall x\ \exists y\ P(x, y)$	[Given]
2.	$\forall x\ P(x, c)$	[$\exists$ Elim: 1, c special]
3.	$\exists z\ \forall x P(x, z)$	[$\exists$ Intro: 2]

(b) Show that $\exists z\ (P(z) \land Q(z))$ follows from $\forall x\ P(x)$ and $\exists y\ Q(y)$.

1.	$\forall x\ P(x)$	[Given]
2.	$\exists y\ Q(y)$	[Given]
3.	Let z be arbitrary	
4.	$P(z)$	[$\forall$ Elim: 1]
5.	$Q(z)$	[$\exists$ Elim: 2, let z be the object that satisfies $Q(z)$]
6.	$P(z) \land Q(z)$	[$\land$ Intro: 4, 5]
7.	$\exists z\ P(z) \land Q(z)$	[$\exists$ Intro: 6]

2. Just The Setup

For each of these statements,

- Translate the sentence into predicate logic.
- Write the first few and last few steps of an inference proof of the statement (you do not need to write the middle – just enough to introduce all givens and assumptions and the conclusion at the end)
- Write the first few sentences and last few sentences of the English proof.

(a) The product of an even integer and an odd integer is even.

(b) There is an integer x s.t. $x^2 > 10$ and $3x$ is even.

(c) For every integer n, there is a prime number p greater than n.

(d) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ for any sets A, B, C.

3. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

(b) $B = \{\{\}, \ \{\{\}\}, \ \{\{\}, \{\}\}, \ \{\{\}, \{\}, \{\}\}, \ \ldots\}$

(c) $C = A \times (B \cup \{7\})$

(d) $D = \varnothing$

(e) $E = \{\varnothing\}$

(f) $F = \mathcal{P}(\{\varnothing\})$

4. Set = Set

Prove the following set identities. Write both a formal inference proof **and** an English proof.

(a) Let the universal set be $\mathcal{U}$. Prove $A \cap \overline{B} \subseteq A \setminus B$ for any sets A, B.

(b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

5. Set Equality

(a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

(b) Let $\mathcal{U}$ be the universal set. Show that $\overline{\overline{X}} = X$.

6. Trickier Set Theory

Note, this problem requires a little more thinking. The solution will cover both the answer as well as the intuition used to arrive at it.

Show that for any set X and any set A such that $A \in \mathcal{P}(X)$, there exists a set $B \in \mathcal{P}(X)$ such that $A \cap B = \emptyset$ and $A \cup B = X$.

7. Predicate Logic Formal Proof

Given $\forall x. T(x) \to M(x)$, we wish to prove $(\exists x. T(x)) \to (\exists y. M(y))$. The following formal proof does this, but it is missing citations for which rules are used, and which lines they are based on. Fill in the blanks with inference rules or predicate logic equivalences, as well as the line numbers.

Then, summarize in English what is going on here.

1. $\forall x. T(x) \to M(x)$ (____________________)
    - 2.1. $\exists x. T(x)$ (____________________)

      Let r be the object that satisfies $T(r)$
    - 2.2. $T(r)$ (____________________, from ______)
    - 2.3. $T(r) \to M(r)$ (____________________, from ______)
    - 2.4. $M(r)$ (____________________, from ______)
    - 2.5. $\exists y. M(y)$ (____________________, from ______)
2. $(\exists x. T(x)) \to (\exists y. M(y))$ (____________________, from ______)

8. Formal Proof (Direct Proof Rule)

Show that $\neg t \to s$ follows from $t \vee q$, $q \to r$ and $r \to s$.

9. Find the Bug

Each of these inference proofs is incorrect. Identify the line (or lines) which incorrectly apply a law, and explain the error. Then, if the claim is false, give concrete examples of propositions to show it is false. If it is true, write a correct proof.

(a) This proof claims to show that given $a \to (b \lor c)$, we can conclude $a \to c$.

1.	$a \to (b \lor c)$	[Given]
2.1.	a	[Assumption]
2.2.	$\neg b$	[Assumption]
2.3.	$b \lor c$	[Modus Ponens, from 1 and 2.1]
2.4.	c	[$\lor$ elimination, from 2.2 and 2.3]
2.	$a \to c$	[Direct Proof Rule, from 2.1-2.4]

(b) This proof claims to show that given $p \to q$ and r, we can conclude $p \to (q \lor r)$.

$1. p \to q$	[Given]
$2. r$	[Given]
$3. p \to (q \lor r)$	[Intro $\lor$ (1,2)]

(c) This proof claims to show that given $p \to q$ and q that we can conclude p

$1. p \to q$	[Given]
$2. q$	[Given]
$3. \neg p \lor q$	[Law of Implication (1)]
$4. p$	[Eliminate $\lor$ (2,3)]

10. A Formal Proof in Predicate Logic

Prove $\exists x\ (P(x) \lor R(x))$ from $\forall x\ (P(x) \lor Q(x))$ and $\forall y\ (\neg Q(y) \lor R(y))$.