Problem Set 2

Due: Wednesday, April 12, by 11:59pm

Instructions

Solutions submission. You must submit your solution via Gradescope. In particular:

- Submit a *single* PDF file containing the solution to all the regular tasks in the homework. *The extra credit is submitted separately in Gradescope - no late days.* Each numbered task should be solved on its own page (or pages). Follow the prompt on Gradescope to link tasks to your pages.

- We encourage you to typeset your solution but do not spend too much time on it; please see the course syllabus. (You can take pictures of hand-written portions and include them as images in the PDF, but they need to be very legible and good contrast.) The homepage provides links to resources to help you doing so using LaTeX. If you do use another tool (e.g., Microsoft Word), we request that you use a proper equation editor to display math (MS Word has one). For example, you should be able to write $\sum_{i=1}^{n} x^i$ instead of `x^1 + x^2 + ... + x^n`. You can also provide a handwritten solution, as long as it is on a single PDF file that satisfies the above submission format requirements. It is your responsibility to make sure handwritten solutions are readable – we will *not* grade unreadable write-ups.

Task 1 – Circuit Breaker [14 pts]

a) Write a table showing the values of the boolean function $B(r, s, t)$ defined as follows. If $r = s = 1$, then $B(r, s, t) = \neg t$. If $r = s = 0$, then $B(r, s, t) = 1$. Finally, if exactly one of r and s is 1, then $B(r, s, t) = s \wedge t$.

b) Create two different circuits that take three input bits, r, s, and t, that each use exactly one OR gate and exactly one XOR gate (and nothing else). The circuits can be whatever you want (not necessarily related to the function B defined in part (a)) subject to those constraints, but they must be different enough that the number of 1s in their output columns differ. For each circuit, draw it as a circuit diagram and give its table of values. The table of values should show the values on each wire in the circuit (not just the inputs and outputs).

c) By using exactly one NOT gate, one OR gate, and one XOR gate (and nothing else), we can make a circuit that calculates the function B from part (a). Give the circuit **and its table of values** (including all wires, not just input and output), the last column of which should match your answer from part (a).

Hint: There are only four places where a NOT gate can be added to a circuit with just one OR and XOR, so there are only a small-ish number of possible circuits to test out. One of them will work.

Task 2 – Short Circuit [14 pts]

a) Write a table showing the values of the boolean function $A(r, s, t)$ defined as follows. If $r = 1$, then $A(r, s, t) = s \rightarrow t$, and if $r = 0$, then $A(r, s, t) = t$.

As always, you should include intermediate expressions (in this case, $s \rightarrow t$) as columns in your table.

b) If we consider circuits with one input, a, that use only a single A gate, there are still $3^3 = 27$ different possibilities because each of the three inputs of the A gate can be either 1, 0, or a.

Find one of those circuits that calculates the same value as $\neg a$. Describe the circuit either by drawing it or writing it as an expression. Then, write a table showing that its values match those of $\neg a$.

Hint: To shorten your search, there is a solution where the three inputs to the A gate are exactly one of each of 1, 0, and a, in some order.

Note: Since there is only one variable, a, your table should have only 2 rows: one for $a = 1$ and one for $a = 0$. In each row, your circuit is calling A with known values for each input, so the reader can figure out the value just by looking at the appropriate row in your table from part (a). For that reason, your table only needs three columns: a, the expression for your circuit, and $\neg a$ for comparison.

c) If we consider circuits with *two inputs*, a and b, that use only a single A gate, there are now $4^3 = 64$ different possibilities because each of the three inputs of the A gate can be either 1, 0, a, or b.

Find one of those circuits that calculates the same value as $a \vee b$. Describe the circuit either by drawing it or writing it as an expression. Then, write a table showing that its values match those of $a \vee b$.

Hint: To shorten your search, there is a solution where the three inputs to the A gate are a, b, and either 1 or 0, in some order.

Note: Your table should now have four rows and four columns since we need to consider all the possible values of both a and b.

d) Combine your circuits from parts (b) and (c) to calculate the same value as $a \wedge b$. Describe the circuit either by drawing it or writing it as an expression. Then, write a table showing that its values match those of $a \wedge b$. (Include intermediate expressions as columns in your table.)

Task 3 – Express Yourself [8 pts]

Consider the following boolean function C:

p	q	r	s	$C(p, q, r, s)$
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

a) Write a Boolean algebra expression for C in sum-of-products form.

b) Write a Boolean algebra expression for C in products-of-sums form.

Task 4 – Keep It Simple [16 pts]

a) Use Boolean algebra identities to simplify your expression from Task 3(a) down to an expression that includes *only 3 gates* (each of which is either AND, OR, or NOT).

You should format your work like an equivalence chain with one expression per line and with the name of the identity applied to produce that line written next to it. However, since we are using Boolean algebra notation, which does not include unnecessary parentheses, you should not include lines that apply Associativity or Commutativity. You can also skip lines that simply apply the Identity rule.

Hint: As we saw in lecture, you may need to temporarily *expand* your expression in order to shrink it fully.

b) Write a truth table for your simplified expression from part (a) and confirm that it matches the one used to define C in Task 3. As always, be sure to include all subexpressions as their own columns.

c) Draw your simplified expression from part (a) as a circuit.

Task 5 – Pet Project [16 pts]

Let the domain of discourse be people and pets. Define the predicates $\text{Person}(x)$ to mean that x is a person and $\text{Pet}(y)$ to mean that y is a pet. Define the predicate $\text{TakesCareOf}(x, y)$ to mean that x takes care of y and the predicate $\text{HasPaws}(y)$ to mean that y has paws.

Translate each of the following logical statements into English. You should not simplify. However, you should use the techniques shown in lecture for producing more natural translations when restricting domains and for avoiding the introduction of variable names when not necessary.

a) $\neg\exists x\,(\text{Person}(x) \land \text{HasPaws}(x))$

b) $\exists x\,\big(\text{Person}(x) \land \exists y\,(\text{Pet}(y) \land \text{TakesCareOf}(x, y) \land \text{TakesCareOf}(y, x))\big)$

c) $\exists x\,\big(\text{Person}(x) \land \exists y\,(\text{Pet}(y) \land \text{TakesCareOf}(x, y) \land \neg\,\text{HasPaws}(y))\big)$

d) $\forall x\,\big(\text{Pet}(x) \to \exists y\,((\text{Person}(y) \lor \text{Pet}(y)) \land \text{TakesCareOf}(y, x))\big)$

Task 6 – Almost Doesn't Count [20 pts]

Let the domain of discourse be people and email messages. Define the predicate $\text{Person}(x)$ to mean that x is a person and the predicate $\text{Message}(y)$ to mean that y is an email. Also define the predicate $\text{Sender}(x, y)$ to mean that x sent message y and the predicate $\text{Receiver}(x, y)$ to mean that x received message y.

Let us also assume an "$=$" operator that is true when x and y are the same person or the same email message. In this problem we will use equality to count (all the way to 2!) in predicate logic, without actually using any numbers in our formulas!

Computer scientists count starting from zero. The formula "$\neg\exists y\ \text{Message}(y)$" can be translated as "there are no messages" or equivalently, "there are zero messages." Hurray, we counted to zero!

a) Use your new-found ability to count to zero in logic to translate the following English sentence.

> Some person has sent zero messages.

Use the predicates from the beginning of this question.

Notice that "Some person..." could also be equivalently phrased as "At least one person...", so actually we also know how to count to (at least) one.

b) Translate this sentence:

> Zero people have sent at least one message.

c) Let's try counting to two, with the following formula.

$$\exists x\,\big(\text{Message}(x) \land \exists y\ \text{Message}(y)\big)$$

This formula fails to count to two: it can be true even when there is only one message. Explain why.

d) We can fix the formula from the previous part by adding $x \neq y$, like this:

$$\exists x \left(\mathsf{Message}(x) \land \exists y \left(\mathsf{Message}(y) \land (x \neq y) \right) \right)$$

We can translate this formula back to English as "There are at least two (different) messages." Use a similar idea to translate the following English sentence into logic:

> Someone has received at least two (different) messages.

e) If we negate the sentence "there are at least two (different) messages", intuitively we should get the sentence "there is at most one message". Show how this works in logic by negating the formula from part d, copied here for your convenience:

$$\exists x \left(\mathsf{Message}(x) \land \exists y \left(\mathsf{Message}(y) \land (x \neq y) \right) \right)$$

Simplify your answer so that it contains no negations. As always, show your work.

Hint: Try to find opportunities to use implication.

Task 7 – Mind Your Ps and Qs [12 pts]

Let P and Q be predicates.

a) Translate the proposition

$$\forall x\, (P(x) \to Q(x))$$

directly into English. This time, *do not* try to make your translation natural sounding. Just do the most literal translation possible.

b) Translate the proposition

$$(\forall x\, P(x)) \to (\forall x\, Q(x))$$

directly into English. Again, *do not* try to make your translation natural sounding. Just do the most literal translation possible.

c) Give an example of predicates P and Q and a domain of discourse so that the propositions from parts a and b **do not** have the same truth value (i.e., one is false and one is true).

d) Give an example of predicates P and Q and a domain of discourse where the propositions from parts a and b **do** have the same truth value (i.e., both are false or both are true).

Task 8 – Extra Credit: Go For the Gold

Five pirates, called Ann, Brenda, Carla, Danielle and Emily, found a treasure of 100 gold coins. On their ship, they decide to split the coins using the following scheme:

- The first pirate in alphabetical order becomes the chief pirate.
- The chief proposes how to share the coins, and all other pirates (except the chief) vote for or against it.
- If 50% or more of the pirates vote for it, then the coins will be shared that way.
- Otherwise, the chief will be thrown overboard, and the process is repeated with the pirates that remain.

Thus, in the first round Ann is the chief: if her proposal is rejected, she is thrown overboard and Brenda becomes the chief, etc; if Ann, Brenda, Carla, and Danielle are thrown overboard, then Emily becomes the chief and keeps the entire treasure.

The pirates' first priority is to stay alive: they will act in such a way as to avoid death. If they can stay alive, they want to get as many coins as possible. Finally, they are a blood-thirsty bunch, if a pirate would get the same number of coins if she voted for or against a proposal, she will vote against so that the pirate who proposed the plan will be thrown overboard.

Assuming that all 5 pirates are intelligent (and aware that all the other pirates are just as aware, intelligent, and bloodthirsty), what will happen? Your solution should indicate which pirates die, and how many coins each of the remaining pirates receives.