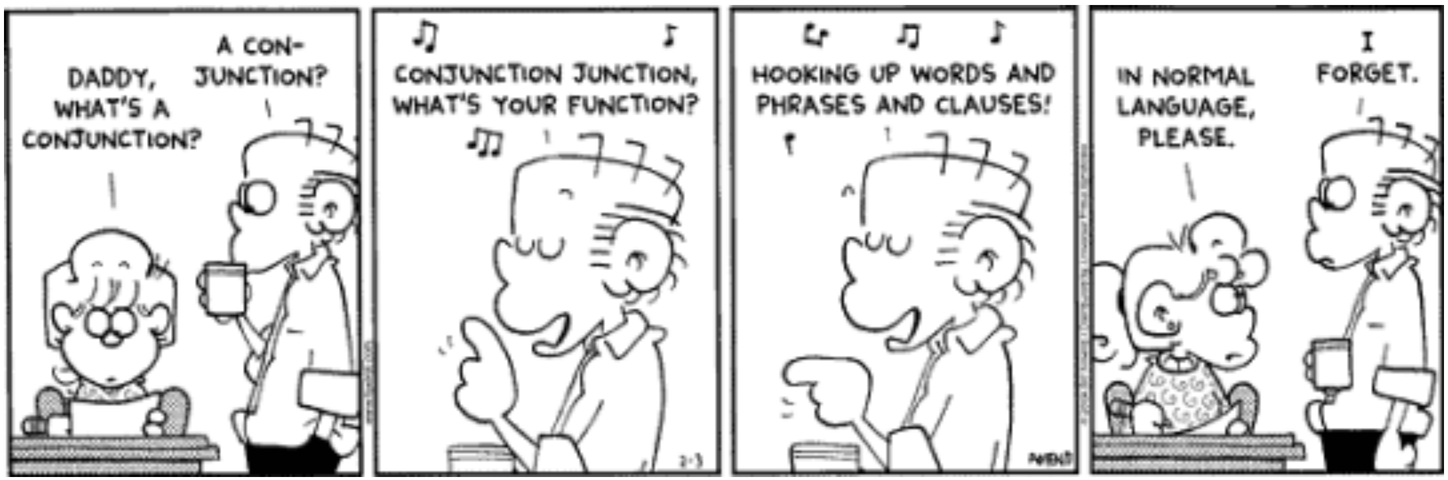
CSE 311: Foundations of Computing

Lecture 2: Logical Equivalence



Last class: Atomic Propositions

Simplest units (words) in this logical language

Propositional Variables: $p, q, r, s, \ldots$

Truth Values:

- **T** for **true**
- **F** for **false**

Last class: Some Connectives & Truth Tables

Negation (not)

p	$\neg p$
T	F
F	T

Conjunction (and)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (or)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Last class: Implication

"If it's raining, then I have my umbrella"

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In English, we can also write

"I have my umbrella if it is raining"

Last class: Truth Table for Vaccine Sentence

$$(\neg r \rightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$$

p	q	r	$\neg r$	$p \wedge q$	$\neg r \longrightarrow (p \wedge q)$	$p \vee q$	$\neg(p \vee q)$	$r \rightarrow \neg(p \vee q)$	$(\neg r \longrightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$
T	T	T	F	T	T	T	F	F	F
T	T	F	T	T	T	T	F	T	T
T	F	T	F	F	T	T	F	F	F
T	F	F	T	F	F	T	F	T	F
F	T	T	F	F	T	T	F	F	F
F	T	F	T	F	F	T	F	T	F
F	F	T	F	F	T	F	T	T	T
F	F	F	T	F	T	F	T	T	T

Last class: Biconditional: $p \leftrightarrow q$

- *p* if and only if *q* (*p* iff *q*)
- *p* is true exactly when *q is true*
- *p* implies *q* and *q* implies *p*
- *p* is necessary and sufficient for *q*

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Converse, Contrapositive

Implication:

$$p \to q$$

Converse:

$$q \to p$$

Contrapositive:

$$\neg q \to \neg p$$

Inverse:

$$\neg p \to \neg q$$

Consider

p: *x* is divisible by 2

q: *x* is divisible by 4

$p \to q$	
$q \to p$	
$\neg q \to \neg p$	
$\neg p \to \neg q$	

Converse, Contrapositive

Implication:

$$p \to q$$

Converse:

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Inverse:

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Consider

p: *x* is divisible by 2

q: *x* is divisible by 4

$p \to q$	F
$q \to p$	T
$\neg q \to \neg p$	F
$\neg p \to \neg q$	T

	Divisible By 2 (p)	Not Divisible By 2 ($\neg p$)
Divisible By 4 (q)	yes	no
Not Divisible By 4 ($\neg q$)	yes	yes

Converse, Contrapositive

Implication:

$p \to q$

Converse:

$q \to p$

Contrapositive:

$\neg q \to \neg p$

Inverse:

$\neg p \to \neg q$

<u>Consider</u>

***p*: *x* is divisible by 2**

***q*: *x* is divisible by 4**

$p \to q$	
$q \to p$	
$\neg q \to \neg p$	
$\neg p \to \neg q$	

	Divisible By 2	Not Divisible By 2
Divisible By 4	4,8,12,...	Impossible
Not Divisible By 4	2,6,10,...	1,3,5,...

Converse, Contrapositive

Implication:

$$p \rightarrow q$$

Contrapositive:

$$\neg q \rightarrow \neg p$$

Converse:

$$q \rightarrow p$$

Inverse:

$$\neg p \rightarrow \neg q$$

Consider

p: x is divisible by 2

q: x is divisible by 4

$p \rightarrow$ q	**F**
$q \rightarrow p$	**T**
$\neg q \rightarrow \neg p$	**F**
$\neg p \rightarrow \neg q$	**T**

	Divisible By 2	Not Divisible By 2
Divisible By 4	4,8,12,...	Impossible
Not Divisible By 4	2,6,10,...	1,3,5,...

Converse, Contrapositive

Implication:

$$p \rightarrow q$$

Converse:

$$q \rightarrow p$$

Contrapositive:

$$\neg q \rightarrow \neg p$$

Inverse:

$$\neg p \rightarrow \neg q$$

How do these relate to each other?

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Converse, Contrapositive

Implication:

$p \to q$

Contrapositive:

$\neg q \to \neg p$

Converse:

$q \to p$

Inverse:

$\neg p \to \neg q$

An implication and it's contrapositive have the same truth value!

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \vee \neg p$ tautology

$p \oplus p$ contradiction

$(p \rightarrow q) \wedge p$

Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \vee \neg p$

This is a tautology. It's called the "law of the excluded middle. If p is true, then $p \vee \neg p$ is true. If p is false, then $p \vee \neg p$ is true.

$p \oplus p$

This is a contradiction. It's always false no matter what truth value p takes on.

$(p \rightarrow q) \wedge p$

This is a contingency. When p is T, q is T, it is true.
When p is F, q is T, it is false.

Logical Equivalence

A = B means A and B are identical "strings":

- $p \wedge q = p \wedge q$
- $p \wedge q \neq q \wedge p$

Logical Equivalence

A = B means **A** and **B** are identical "strings":

- $p \wedge q = p \wedge q$

  These are equal, because they are character-for-character identical.

- $p \wedge q \neq q \wedge p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

A ≡ B means **A** and **B** have identical truth values:

- $p \wedge q \equiv p \wedge q$
- $p \wedge q \equiv q \wedge p$
- $p \wedge q \not\equiv q \vee p$

Logical Equivalence

A = B means A and B are identical "strings":

- $p \wedge q = p \wedge q$

  These are equal, because they are character-for-character identical.

- $p \wedge q \neq q \wedge p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

A ≡ B means A and B have identical truth values:

- $p \wedge q \equiv p \wedge q$

  Two formulas that are **equal** also are equivalent.

- $p \wedge q \equiv q \wedge p$

  These two formulas have the same truth table!

- $p \wedge q \not\equiv q \vee p$

  When p=T and q=F, $p \wedge q$ is false, but $p \vee q$ is true!

$A \leftrightarrow B$ vs. $A \equiv B$

$A \leftrightarrow B$ is a **proposition** that may be true or false depending on the truth values of A and B.

$A \equiv B$ is an **assertion** over all possible truth values that A and B always have the same truth values.

$A \equiv B$ and $(A \leftrightarrow B) \equiv T$ have the same meaning as does "$A \leftrightarrow B$ is a tautology"

Logical Equivalence $A \equiv B$

$A \equiv B$ is an assertion that ***two propositions*** A and B always have the same truth values.

$A \equiv B$ and $(A \leftrightarrow B) \equiv T$ have the same meaning.

$p \wedge q \equiv q \wedge p$

p	q	$p \wedge q$	$q \wedge p$	$(p \wedge q) \leftrightarrow (q \wedge p)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

$p \wedge q \not\equiv q \vee p$

When p is **T** and q is **F**, $p \wedge q$ is false, but $q \vee p$ is true

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Negate the statement:

"My code compiles or there is a bug."

To negate the statement,

ask "when is the original statement false".

It's false when not(my code compiles) AND not(there is a bug).

Translating back into English, we get:

My code doesn't compile and there is not a bug.

De Morgan's Laws

Example: $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$\neg(p \wedge q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

```
if (!(front != null && value > front.data)) {
    front = new ListNode(value, front);
} else {
    ListNode current = front;
    while (current.next != null && current.next.data < value))
        current = current.next;
    current.next = new ListNode(value, current.next);
}
```

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

```
!(front != null && value > front.data)
```

$$\equiv$$

```
front == null || value <= front.data
```

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Law of Implication

$$p \to q \equiv \neg p \vee q$$

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Some Familiar Properties of Arithmetic

- $x + y = y + x$ **(Commutativity)**
- $x \cdot (y + z) = x \cdot y + x \cdot z$ **(Distributivity)**
- $(x + y) + z = x + (y + z)$ **(Associativity)**

Important Equivalences

- Identity
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- Domination
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- Commutative
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- Associative
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Absorption
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- Negation
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

Some Familiar Properties of Arithmetic

- $x \cdot 1 = x$ **(Identity)**
- $x + 0 = x$

- $x \cdot 0 = 0$ **(Domination)**

Important Equivalences

- Identity
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- Domination
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- Commutative
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- Associative
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Absorption
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- Negation
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

Some Familiar Properties of Arithmetic

- Usual properties hold under relabeling:
  - 0, 1 becomes F, T
  - "+" becomes "$\vee$"
  - " $\cdot$ " becomes "$\wedge$"
- But there are some new facts:
  - Distributivity works for both "$\wedge$" and "$\vee$"
  - Domination works with T
- There are some other facts specific to logic…

Important Equivalences

- Identity
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- Domination
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- Commutative
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- Associative
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Absorption
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- Negation
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

Important Equivalences

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

Using Equivalences

- Note that p, q, and r can be **any** propositions (not just atomic propositions)

- Ex: $(r \rightarrow s) \wedge (\neg t) \equiv (\neg t) \wedge (r \rightarrow s)$
  - apply commutativity: $p \wedge q \equiv q \wedge p$
    with $p := r \rightarrow s$
    and $q := \neg t$

One more easy equivalence

Double Negation

$$p \equiv \neg\neg p$$

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

Understanding logic and circuits

When do two logic formulas mean the same thing?

What logical properties can we infer from other ones?

Basic rules of reasoning and logic

- Working with logical formulas
  - Simplification
  - Testing for equivalence

- Applications
  - Query optimization
  - Search optimization and caching
  - Artificial Intelligence
  - Program verification

Computing Equivalence

Given two propositions, can we write an algorithm to determine if they are equivalent?

What is the runtime of our algorithm?

Computing Equivalence

Given two propositions, can we write an algorithm to determine if they are equivalent?

Yes! Generate the truth tables for both propositions and check if they are the same for every entry.

What is the runtime of our algorithm?

Every atomic proposition has two possibilities (T, F). If there are n atomic propositions, there are 2^n rows in the truth table.

Another approach: Logical Proofs

To show A is equivalent to B

- Apply a series of logical equivalences to sub-expressions to convert A to B

To show A is a tautology

- Apply a series of logical equivalences to sub-expressions to convert A to **T**

Another approach: Logical Proofs

To show A is equivalent to B

– Apply a series of logical equivalences to sub-expressions to convert A to B

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general proof looks like:

$$p \vee (p \wedge p) \equiv (\qquad\qquad\qquad\qquad)$$
$$\equiv p$$

Another approach: Logical Proofs

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general proof looks like:

$$p \vee (p \wedge p) \equiv (\quad p \vee p \quad) \quad \text{Idem.}$$
$$\equiv p \quad \text{Idem}$$

Logical Proofs

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
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  - $p \wedge p \equiv p$
- **Commutative**
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  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
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  - $p \vee \neg p \equiv \mathrm{T}$
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De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \to q \equiv \neg p \vee q$$

Contrapositive

$$p \to q \equiv \neg q \to \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general proof looks like:

$$p \vee (p \wedge p) \equiv (\quad p \vee p \quad) \qquad \text{Idempotent}$$
$$\equiv p \qquad \text{Idempotent}$$

Logical Proofs

To show A is a tautology

- Apply a series of logical equivalences to sub-expressions to convert A to **T**

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general proof looks like:

$$\begin{aligned} \neg p \vee (p \vee p) &\equiv (\qquad\qquad\qquad\qquad) \\ &\equiv (\qquad\qquad\qquad\qquad) \\ &\equiv \mathbf{T} \end{aligned}$$

Logical Proofs

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \to q \equiv \neg p \vee q$$

Contrapositive

$$p \to q \equiv \neg q \to \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general proof looks like:

$$\neg p \vee (p \vee p) \equiv (\ \neg p \vee p\ ) \quad \text{idem.}$$
$$\equiv (\ p \vee \neg p\ ) \quad \text{comm.}$$
$$\equiv \mathbf{T} \quad \text{negation}$$

Logical Proofs

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general proof looks like:

$$\neg p \vee (p \vee p) \equiv (\quad \neg p \vee p \quad) \quad \text{Idempotent}$$
$$\equiv (\quad p \vee \neg p \quad) \quad \text{Commutative}$$
$$\equiv \mathbf{T} \quad \text{Negation}$$

Prove these propositions are equivalent: Option 1

Prove: $p \wedge (p \rightarrow q) \equiv p \wedge q$

Make a Truth Table and show:

$$(p \wedge (p \rightarrow q)) \leftrightarrow (p \wedge r) \equiv \mathbf{T}$$

p	r	$p \rightarrow r$	$(p \wedge (p \rightarrow r))$	$p \wedge r$	$(p \wedge (p \rightarrow r)) \leftrightarrow (p \wedge r)$
T	T	T	T	T	T
T	F	F	F	F	T
F	T	T	F	F	T
F	F	T	F	F	T

Prove these propositions are equivalent: Option 2

Prove: $p \wedge (p \rightarrow q) \equiv p \wedge q$

$$p \wedge (p \rightarrow q) \equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv p \wedge q$$

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg \neg p$$

Prove these propositions are equivalent: Option 2

Prove: $p \wedge (p \rightarrow q) \equiv p \wedge q$

$p \wedge (p \rightarrow q) \equiv p \wedge (\neg p \vee q)$	Law of Implication
$\equiv (p \wedge \neg p) \vee (p \wedge q)$	Distributive
$\equiv \mathbf{F} \vee (p \wedge q)$	Negation
$\equiv (p \wedge q) \vee \mathbf{F}$	Commutative
$\equiv p \wedge q$	Identity

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg \neg p$$

Prove this is a Tautology: Option 1

$$(p \wedge r) \rightarrow (r \vee p)$$

Make a Truth Table and show:

$$(p \wedge r) \rightarrow (r \vee p) \equiv \mathrm{T}$$

$\boldsymbol{p}$	$\boldsymbol{r}$	$\boldsymbol{p \wedge r}$	$\boldsymbol{r \vee p}$	$\boldsymbol{(p \wedge r) \rightarrow (r \vee p)}$
T	**T**			
T	**F**			
F	**T**			
F	**F**			

Prove this is a Tautology: Option 1

$$(p \land r) \to (r \lor p)$$

Make a Truth Table and show:

$$(p \land r) \to (r \lor p) \equiv \mathbf{T}$$

p	r	$p \land r$	$r \lor p$	$(p \land r) \to (r \lor p)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Prove this is a Tautology: Option 2

$$(p \land r) \rightarrow (r \lor p)$$

Use a series of equivalences like so:

$$(p \land r) \rightarrow (r \lor p) \equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv \quad \mathbf{T}$$

Associative
- $(p \lor q) \lor r \equiv p \lor (q \lor r)$
- $(p \land q) \land r \equiv p \land (q \land r)$

Distributive
- $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
- $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

Absorption
- $p \lor (p \land q) \equiv p$
- $p \land (p \lor q) \equiv p$

Negation
- $p \lor \neg p \equiv \mathrm{T}$
- $p \land \neg p \equiv \mathrm{F}$

Identity
- $p \land \mathrm{T} \equiv p$
- $p \lor \mathrm{F} \equiv p$

Domination
- $p \lor \mathrm{T} \equiv \mathrm{T}$
- $p \land \mathrm{F} \equiv \mathrm{F}$

Idempotent
- $p \lor p \equiv p$
- $p \land p \equiv p$

Commutative
- $p \lor q \equiv q \lor p$
- $p \land q \equiv q \land p$

Use a series of equivalences like so:

$$
\begin{aligned}
(p \land r) \rightarrow (r \lor p) &\equiv \neg(p \land r) \lor (r \lor p) \\
&\equiv (\neg p \lor \neg r) \lor (r \lor p) \\
&\equiv \neg p \lor (\neg r \lor (r \lor p)) \\
&\equiv \neg p \lor ((\neg r \lor r) \lor p) \\
&\equiv \neg p \lor (p \lor (\neg r \lor r)) \\
&\equiv (\neg p \lor p) \lor (\neg r \lor r) \\
&\equiv (p \lor \neg p) \lor (r \lor \neg r) \\
&\equiv \mathrm{T} \lor \mathrm{T} \\
&\equiv \mathrm{T}
\end{aligned}
$$

Prove this is a Tautology: Option 2

$$(p \land r) \rightarrow (r \lor p)$$

Use a series of equivalences like so:

$(p \land r) \rightarrow (r \lor p) \equiv \neg(p \land r) \lor (r \lor p)$	Law of Implication
$\equiv (\neg p \lor \neg r) \lor (r \lor p)$	De Morgan
$\equiv \neg p \lor (\neg r \lor (r \lor p))$	Associative
$\equiv \neg p \lor ((\neg r \lor r) \lor p)$	Associative
$\equiv \neg p \lor (p \lor (\neg r \lor r))$	Commutative
$\equiv (\neg p \lor p) \lor (\neg r \lor r)$	Associative
$\equiv (p \lor \neg p) \lor (r \lor \neg r)$	Commutative (twice)
$\equiv \mathbf{T} \lor \mathbf{T}$	Negation (twice)
$\equiv \mathbf{T}$	Domination/Identity

Identity
- $p \land \mathrm{T} \equiv p$
- $p \lor \mathrm{F} \equiv p$

Domination
- $p \lor \mathrm{T} \equiv \mathrm{T}$
- $p \land \mathrm{F} \equiv \mathrm{F}$

Idempotent
- $p \lor p \equiv p$
- $p \land p \equiv p$

Commutative
- $p \lor q \equiv q \lor p$
- $p \land q \equiv q \land p$

Logical Proofs of Equivalence/Tautology

- Not smaller than truth tables when there are only a few propositional variables...
- ...but usually ***much shorter*** than truth table proofs when there are many propositional variables
- A big advantage will be that we can extend them to a more in-depth understanding of logic for which truth tables don't apply.