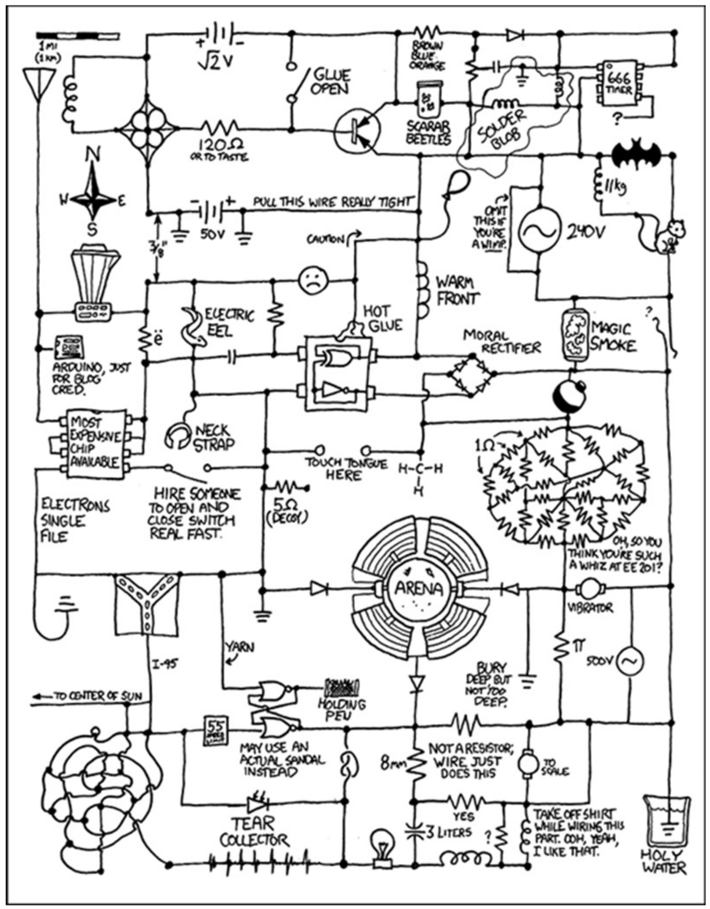
CSE 311: Foundations of Computing

Lecture 4: Boolean Algebra, Circuits, Canonical Forms

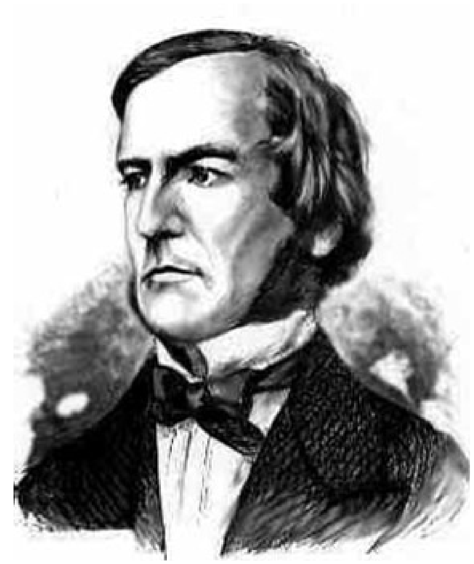


OH 2:30
today
CSE 368

Last Time: Boolean Algebra

- Usual notation used in circuit design

- Boolean algebra
  OR AND
  - a set of elements B containing {0, 1}
  - binary operations { + , • } "COMPLEMENT"
  - and a unary operation { ’ } (NOT)
  - such that the following axioms hold:



For any a, b, c in B:

1. closure:	a + b is in B	a • b is in B
2. commutativity:	a + b = b + a	a • b = b • a
3. associativity:	a + (b + c) = (a + b) + c	a • (b • c) = (a • b) • c
4. distributivity:	a + (b • c) = (a + b) • (a + c)	a • (b + c) = (a • b) + (a • c)
5. identity:	a + 0 = a	a • 1 = a
6. complementarity:	a + a’ = 1	a • a’ = 0
7. null:	a + 1 = 1	a • 0 = 0
8. idempotency:	a + a = a	a • a = a
9. involution:	(a’)’ = a	

Warm-up Exercise

- Create a Boolean Algebra expression for C below in terms of the variables $\boldsymbol{a}$ and $\boldsymbol{b}$

$C = 1$

$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{C(a,b)}$
1	1	0
1	0	1
0	1	1
0	0	0

~~ab~~

$a \cdot b'$

$a' \cdot b$

$a \cdot b' + a' \cdot b$

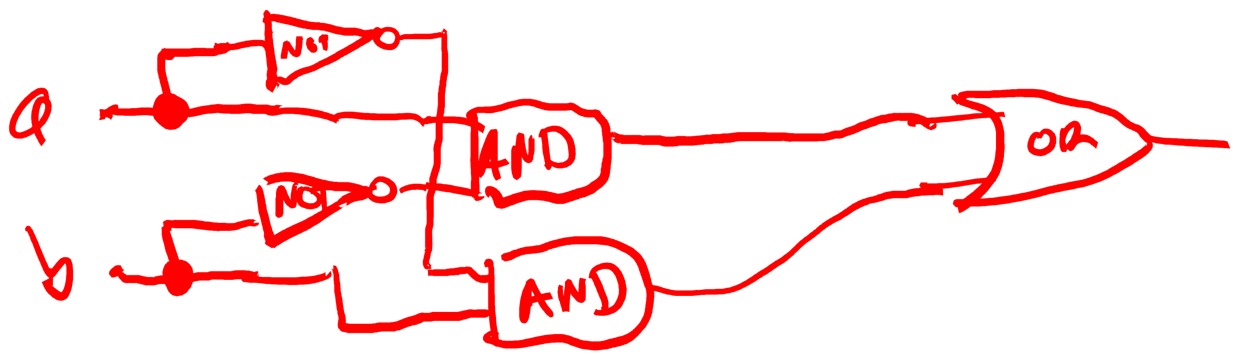
$$ab' + a'b$$

Warm-up Exercise

- Create a Boolean Algebra expression for "c" below in terms of the variables a and b

$$c = ab' + a'b$$

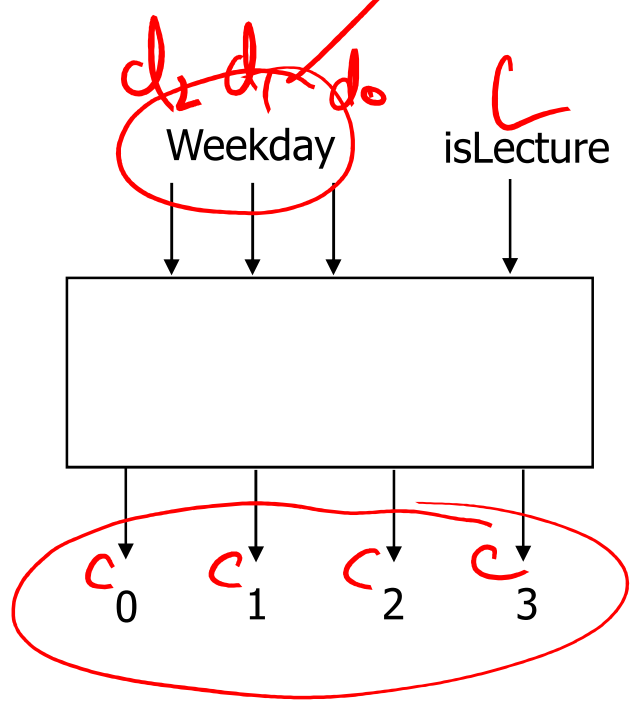
- Draw this as a circuit (using AND, OR, NOT)



Last Time: Combinational Logic

Encoding:

- Binary number for weekday (Binary encoding)
- One bit for each possible output ("1-Hot" encoding)



Last Time: Truth Table to Logic

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$d_2' \cdot d_1' \cdot d_0' \cdot L$

$d_2' \cdot d_1' \cdot d_0 \cdot L$

Either situation causes c_3 to be true. So, we "or" them.

$$c_3 = ((d_2' \cdot d_1') \cdot d_0') \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

Last Time: Truth Table to Logic

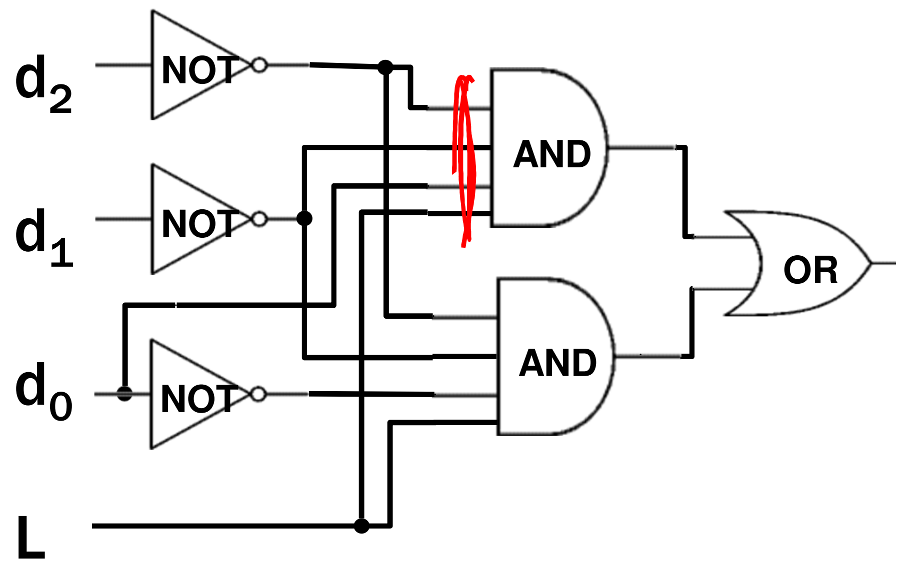
$c_0 = d_2 \cdot d_1' \cdot d_0 \cdot L' + d_2 \cdot d_1 \cdot d_0' + d_2 \cdot d_1 \cdot d_0$

$c_1 = d_2' \cdot d_1' \cdot d_0' \cdot L' + d_2' \cdot d_1' \cdot d_0 \cdot L' + d_2' \cdot d_1 \cdot d_0' \cdot L' + d_2' \cdot d_1 \cdot d_0 \cdot L' + d_2 \cdot d_1' \cdot d_0' + d_2 \cdot d_1' \cdot d_0 \cdot L$

$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$

$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$

Here's c_3 as a circuit:



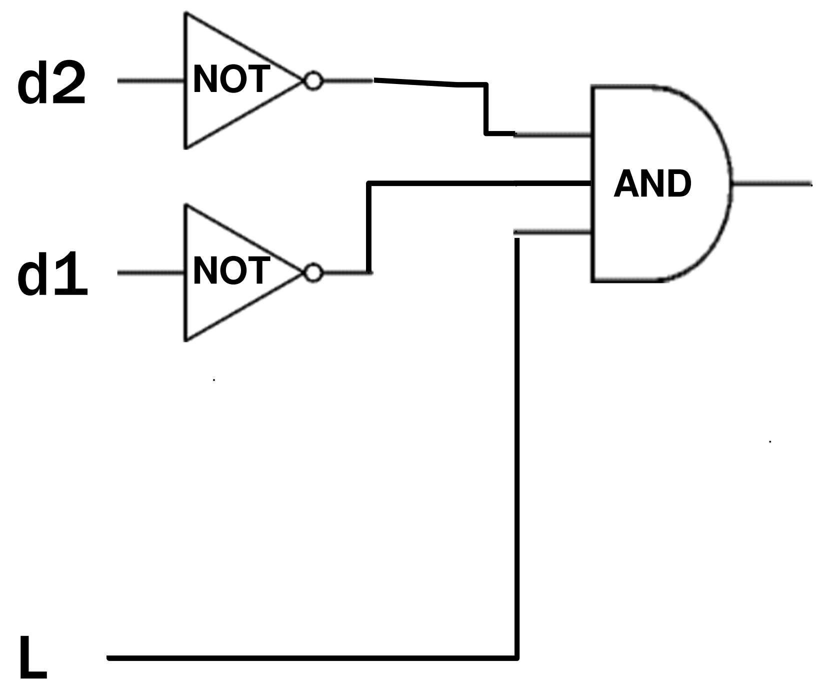
Simplifying using Boolean Algebra

c3 = d2’•d1’•d0’•L + d2’•d1’•d0•L

= d2’•d1’•(d0’ + d0)•L

= d2’•d1’•1•L

= d2’•d1’•L



Important Corollaries of this Construction

- **$\neg$, $\wedge$, $\vee$ can implement any Boolean function**

  we didn't need any others to do this

- **Actually, just $\neg$, $\wedge$ (or $\neg$, $\vee$) are enough**

  follows by De Morgan's laws

$a \vee b$

$\neg(\neg a \wedge \neg b)$

$=$

$(\neg\neg a \vee \neg\neg b)$

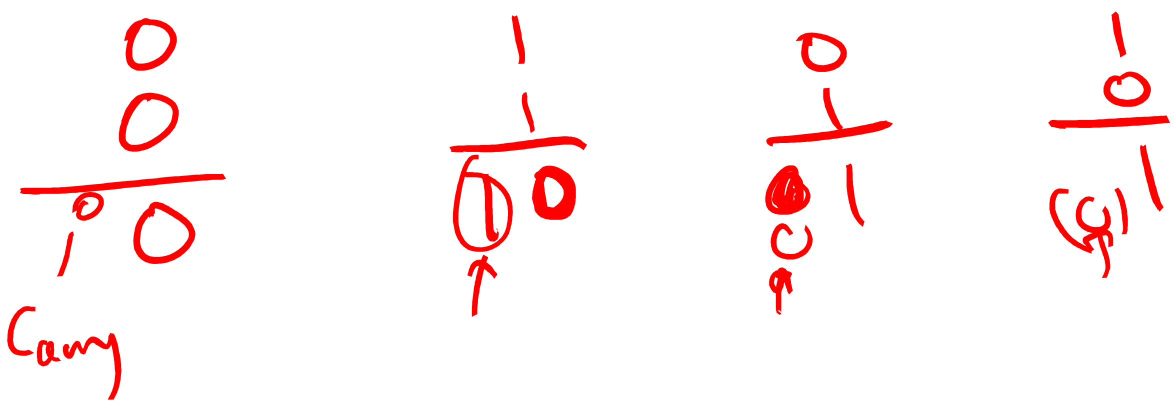
$=$

$(a \vee b)$

- **Actually, just NAND (or NOR)**

1-bit Binary Adder

A	$0 + 0 = 0$ (with $C_{OUT} = 0$)
$\underline{+\ B}$	$0 + 1 = 1$ (with $C_{OUT} = 0$)
S	$1 + 0 = 1$ (with $C_{OUT} = 0$)
(C_{OUT})	$1 + 1 = 0$ (with $C_{OUT} = 1$)



1-bit Binary Adder

A	$0 + 0 = 0$ (with $C_{OUT} = 0$)
$+\ B$	$0 + 1 = 1$ (with $C_{OUT} = 0$)
S	$1 + 0 = 1$ (with $C_{OUT} = 0$)
(C_{OUT})	$1 + 1 = 0$ (with $C_{OUT} = 1$)

Idea: chain these together to add larger numbers

Recall from elementary school:

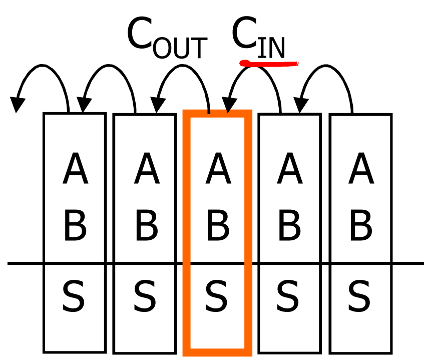
$$\begin{array}{r} {}^{1}\,{}^{1} \\ 2\,4\,8 \\ +\ 3\,7\,5 \\ \hline 6\,2\,3 \end{array}$$

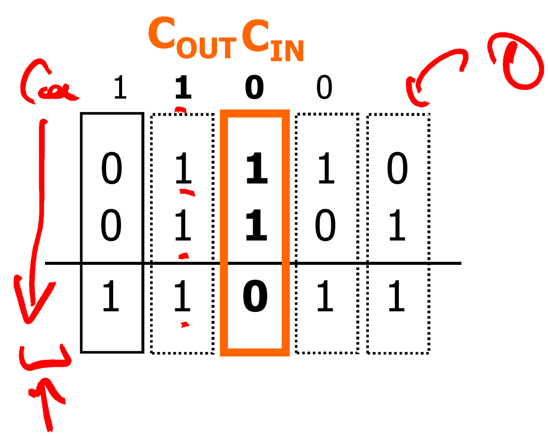
1-bit Binary Adder

A	0 + 0 = 0 (with C_{OUT} = 0)
+ B	0 + 1 = 1 (with C_{OUT} = 0)
S	1 + 0 = 1 (with C_{OUT} = 0)
(C_{OUT})	1 + 1 = 0 (with C_{OUT} = 1)

Idea: These are chained together with a carry-in

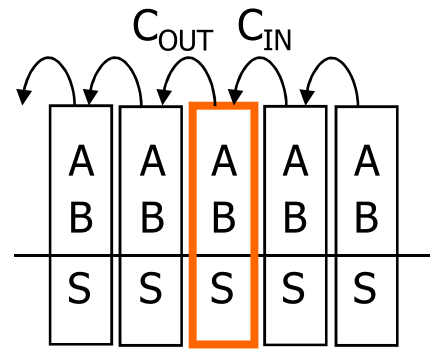
(C_{IN})
A
+ B
S Sum
(C_{OUT})





1-bit Binary Adder

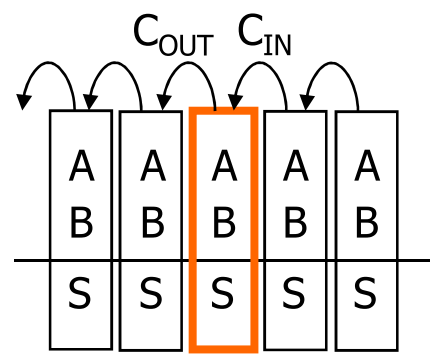
- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Derive an expression for S

$A' \cdot B' \cdot C_{IN}$

$A' \cdot B \cdot C_{IN}'$

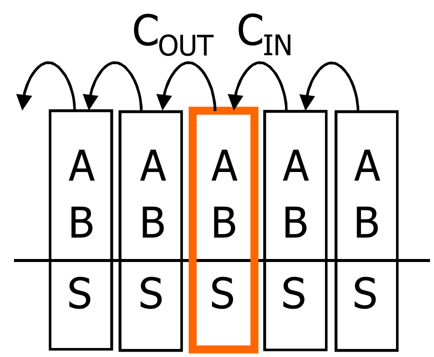
$A \cdot B' \cdot C_{IN}'$

$A \cdot B \cdot C_{IN}$

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Derive an expression for C_{OUT}

$A' \cdot B \cdot C_{IN}$

$A \cdot B' \cdot C_{IN}$

$A \cdot B \cdot C_{IN}'$

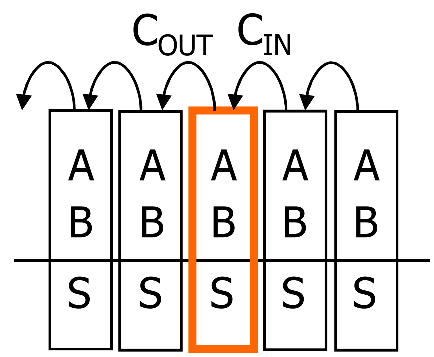
$A \cdot B \cdot C_{IN}$

$$C_{OUT} = A' \cdot B \cdot C_{IN} + A \cdot B' \cdot C_{IN} + A \cdot B \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

$$S = A' \bullet B' \bullet C_{IN} + A' \bullet B \bullet C_{IN}' + A \bullet B' \bullet C_{IN}' + A \bullet B \bullet C_{IN}$$

$$C_{OUT} = A' \bullet B \bullet C_{IN} + A \bullet B' \bullet C_{IN} + A \bullet B \bullet C_{IN}' + A \bullet B \bullet C_{IN}$$

Apply Theorems to Simplify Expressions

The theorems of Boolean algebra can simplify expressions

– e.g., full adder's carry-out function

$a = a + a$

$$
\begin{aligned}
\text{Cout} &= A'\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= A'\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin + A\,B\,Cin \\
&= A'\,B\,Cin + A\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= (A' + A)\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= (1)\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin + A\,B\,Cin \\
&= B\,Cin + A\,B'\,Cin + A\,B\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,(B' + B)\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,(1)\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,Cin + A\,B\,(Cin' + Cin) \\
&= B\,Cin + A\,Cin + A\,B\,(1) \\
&= B\,Cin + A\,Cin + A\,B
\end{aligned}
$$

"Karnaugh map"

Apply Theorems to Simplify Expressions

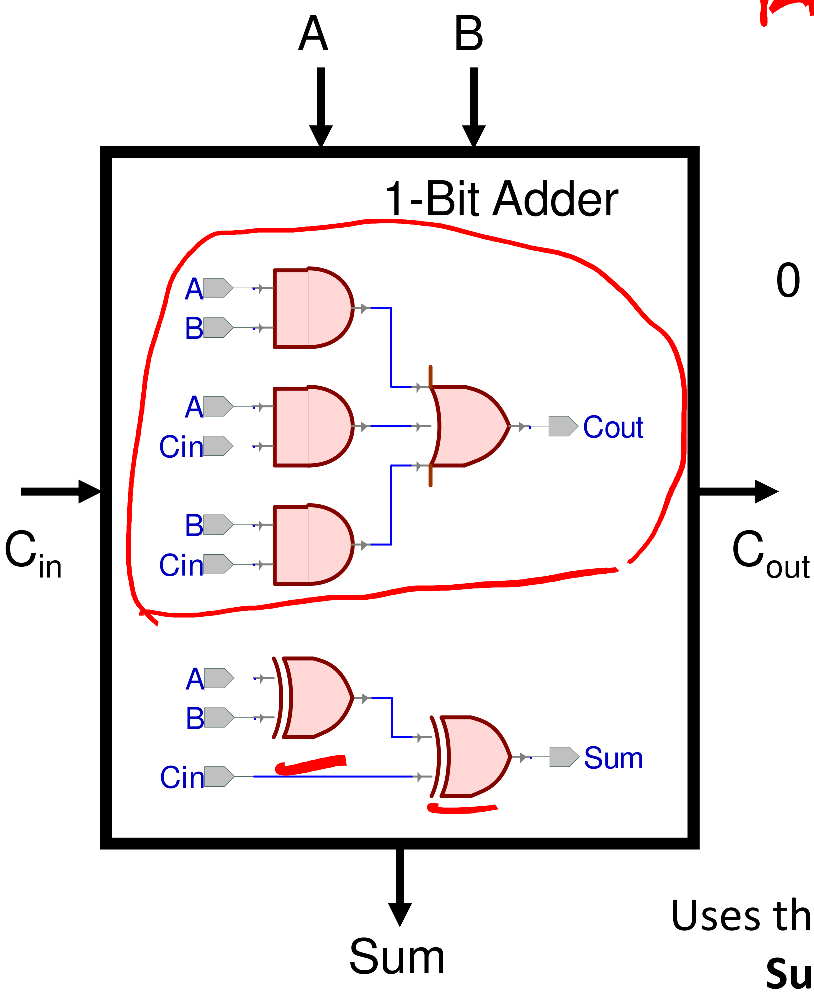
The theorems of Boolean algebra can simplify expressions

- **e.g., full adder's carry-out function**

$$
\begin{aligned}
\text{Cout} &= A'\,B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= A'\,B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + \boxed{A\,B\,\text{Cin} + A\,B\,\text{Cin}} \\
&= A'\,B\,\text{Cin} + A\,B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= (A' + A)\,B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= (1)\,B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + \boxed{A\,B\,\text{Cin} + A\,B\,\text{Cin}} \\
&= B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= B\,\text{Cin} + A\,(B' + B)\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= B\,\text{Cin} + A\,(1)\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= B\,\text{Cin} + A\,\text{Cin} + A\,B\,(\text{Cin}' + \text{Cin}) \\
&= B\,\text{Cin} + A\,\text{Cin} + A\,B\,(1) \\
&= B\,\text{Cin} + A\,\text{Cin} + A\,B
\end{aligned}
$$

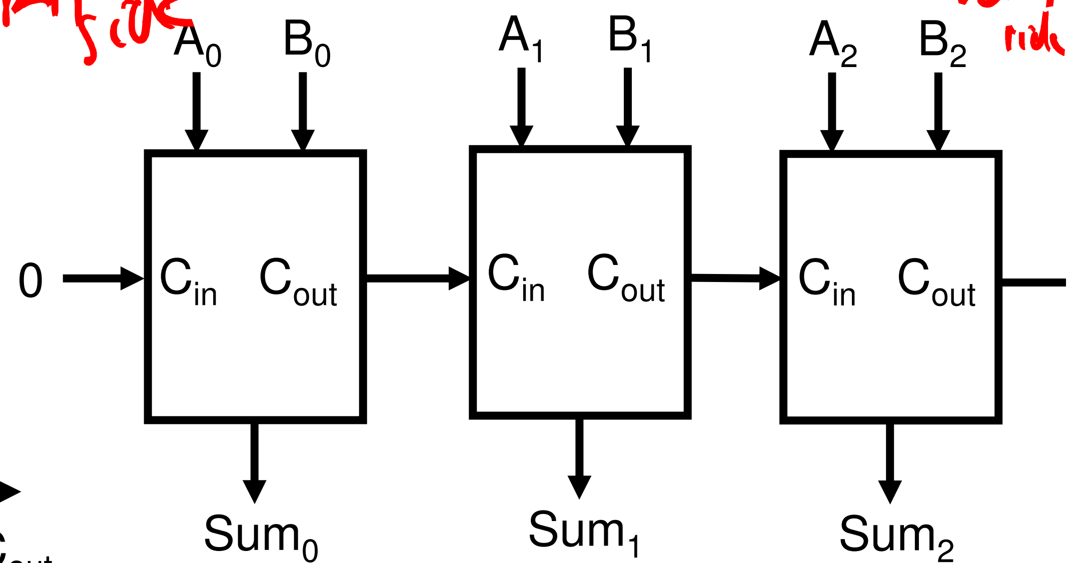
adding extra terms
creates new factoring
opportunities

A 2-bit Ripple-Carry Adder



Right hand side

left hand side



Uses the fact that

$$\mathbf{Sum} = A' \bullet B' \bullet C_{IN} + A' \bullet B \bullet C_{IN}' + A \bullet B' \bullet C_{IN}' + A \bullet B \bullet C_{IN}$$

is equivalent to $\mathbf{Sum} = (A \oplus B) \oplus C_{IN}$

Mapping Truth Tables to Logic Gates

Given a truth table:

1. Write the output in a table
2. Write the Boolean expression
3. Minimize the Boolean expression
4. Draw as gates
5. Map to available gates

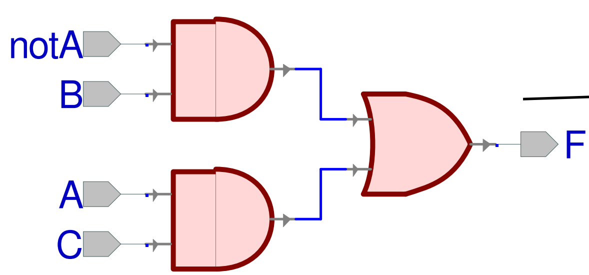
A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

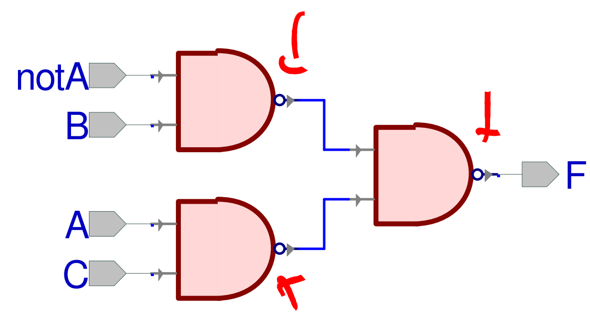
$F = A'BC' + A'BC + AB'C + ABC$

$= A'B(C'+C) + AC(B'+B)$

$= A'B + AC$

$((a \cdot b)' \cdot (c \cdot d)' \cdot (e \cdot f)')'$





Canonical Forms

- Truth table is the unique signature of a 0/1 function
- The same truth table can have many gate realizations
  - We've seen this already
  - Depends on how good we are at Boolean simplification
- Canonical forms
  - Standard forms for a Boolean expression
  - We all produce the same expression

Sum-of-Products Canonical Form

- AKA Disjunctive Normal Form (DNF)
- AKA Minterm Expansion

③ Add the minterms together

$F = A'B'C + A'BC + AB'C + ABC' + ABC'$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

① Read T rows off truth table

② Convert to Boolean Algebra

001 → $A'B'C$

011 → $A'BC$

101 → $AB'C$

110 → ABC'

111 → ABC

F

Sum-of-Products Canonical Form

Product term (or minterm)

- ANDed product of literals – input combination for which output is true
- each variable appears exactly once, true or inverted (but not both)

A	B	C	minterms
0	0	0	A'B'C'
0	0	1	A'B'C
0	1	0	A'BC'
0	1	1	A'BC
1	0	0	AB'C'
1	0	1	AB'C
1	1	0	ABC'
1	1	1	ABC

F in canonical form:

$$F(A, B, C) = A'B'C + A'BC + AB'C + ABC' + ABC$$

canonical form ≠ minimal form

$$\begin{aligned} F(A, B, C) &= A'B'C + A'BC + AB'C + ABC + ABC' \\ &= (A'B' + A'B + AB' + AB)C + ABC' \\ &= ((A' + A)(B' + B))C + ABC' \\ &= C + ABC' \\ &= ABC' + C \\ &= AB + C \end{aligned}$$

Product-of-Sums Canonical Form

- AKA Conjunctive Normal Form (CNF)
- AKA Maxterm Expansion

4 Multiply the maxterms together

F =

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

1 Read F rows off truth table

2 Negate all bits

3 Convert to Boolean Algebra

F

Product-of-Sums Canonical Form

- AKA Conjunctive Normal Form (CNF)
- AKA Maxterm Expansion

clause

④ Multiply the maxterms together

$F = (A + B + C)(A + B' + C)(A' + B + C)$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

① Read F rows off truth table	② Negate all bits	③ Convert to Boolean Algebra
000 →	111 →	$A + B + C$
010 →	101 →	$A + B' + C$
100 →	011 →	$A' + B + C$

F

A+B+C

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F’)’ = F
- We know how to get a minterm expansion for F’

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F' = A'B'C' + A'BC' + AB'C'$$

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F')' = F
- We know how to get a minterm expansion for F'

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F' = A'B'C' + A'BC' + AB'C'$$

Taking the complement of both sides…

$$(F')' = (A'B'C' + A'BC' + AB'C')'$$

And using DeMorgan/Comp….

$$F = (A'B'C')' \ (A'BC')' \ (AB'C')'$$

$$F = (A + B + C)(A + B' + C)(A' + B + C)$$

Product-of-Sums Canonical Form

Sum term (or maxterm)

- ORed sum of literals – input combination for which output is false
- each variable appears exactly once, true or inverted (but not both)

A	B	C	maxterms
0	0	0	$A+B+C$
0	0	1	$A+B+C'$
0	1	0	$A+B'+C$
0	1	1	$A+B'+C'$
1	0	0	$A'+B+C$
1	0	1	$A'+B+C'$
1	1	0	$A'+B'+C$
1	1	1	$A'+B'+C'$

F in canonical form:

$$F(A, B, C) = (A + B + C)\,(A + B' + C)\,(A' + B + C)$$

canonical form ≠ minimal form

$$\begin{aligned} F(A, B, C) &= (A + B + C)\,(A + B' + C)\,(A' + B + C) \\ &= (A + B + C)\,(A + B' + C) \\ &\quad\ (A + B + C)\,(A' + B + C) \\ &= (A + C)\,(B + C) \end{aligned}$$