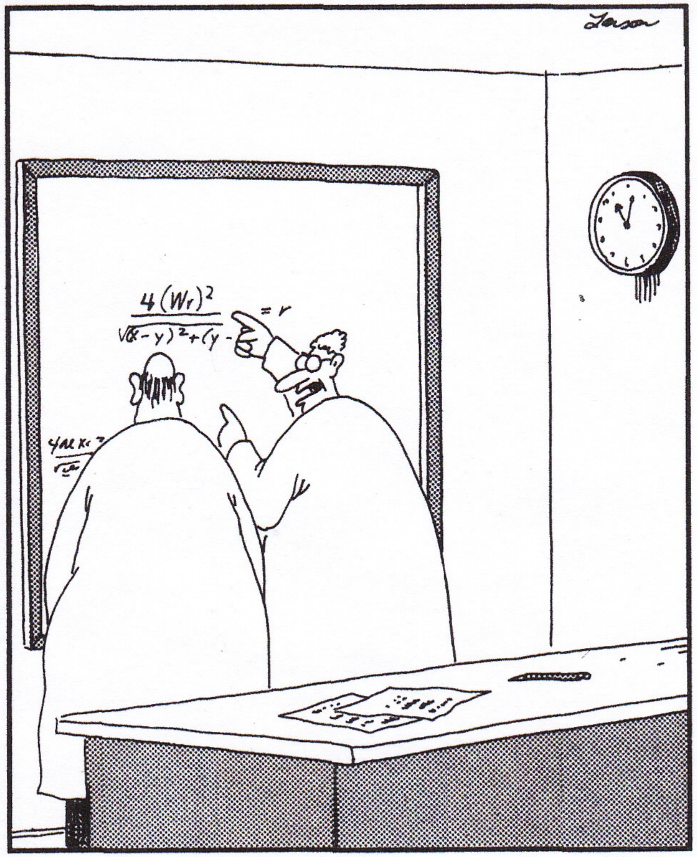
CSE 311: Foundations of Computing

Lecture 9: English Proofs, Strategies & Number Theory



"Yes, yes, I *know* that, Sidney…*everybody* knows *that!*…But look: Four wrongs *squared,* minus two wrongs to the fourth power, divided by this formula, *do* make a right."

Last class: Inference Rules for Quantifiers

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore \quad P(a) \quad \textbf{(for any } a\text{)}}$$

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \textbf{ for some } \textbf{\textit{special}}{*}{*}\ c}$$

Intro ∀

$$\frac{\text{“}\textbf{Let } a \textbf{ be arbitrary}^{*}\text{”}\ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

** c is a NEW name.
List all dependencies for c.

* in the domain of P. No other name in P depends on a.

dependencies:
other named arbitrary constants in ∃x P(x)

Last class: Formal & English Proofs: Even and Odd

$\mathrm{Even}(x) \equiv \exists y\ (x=2y)$
$\mathrm{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The sum of two odd numbers is even."

Let x and y be arbitrary integers.

Suppose that both are odd.

Then, we have x = 2a+1 for some integer a and y = 2b+1 for some integer b.

Their sum is x+y = ... = 2(a+b+1)

so x+y is, by definition, even.

Since x and y were arbitrary, the sum of two odd integers is even.

1.	Let **x** be an arbitrary integer	
2.	Let **y** be an arbitrary integer	
3.1	$\mathrm{Odd}(x) \land \mathrm{Odd}(y)$	Assumption
3.2	$\mathrm{Odd}(x)$	Elim ∧: 2.1
3.3	$\mathrm{Odd}(y)$	Elim ∧: 2.1
3.4	$\exists z\ (x = 2z+1)$	Def of Odd: 2.2
3.5	$x = 2a+1$	Elim ∃: 2.4: **a** depend **x**
3.6	$\exists z\ (y = 2z+1)$	Def of Odd: 2.3
3.7	$y = 2b+1$	Elim ∃: 2.5: **b** depend **y**
3.8	$x+y = 2(a+b+1)$	Algebra
3.9	$\exists z\ (x+y = 2z)$	Intro ∃: 2.4
3.10	$\mathrm{Even}(x+y)$	Def of Even
3.	$(\mathrm{Odd}(x) \land \mathrm{Odd}(y)) \rightarrow \mathrm{Even}(x+y)$	DPR
4.	$\forall y\ ((\mathrm{Odd}(x) \land \mathrm{Odd}(y)) \rightarrow \mathrm{Even}(x+y))$	Intro ∀
5.	$\forall x \forall y\ ((\mathrm{Odd}(x) \land \mathrm{Odd}(y)) \rightarrow \mathrm{Even}(x+y))$	Intro ∀

Last class: Even and Odd

Predicate Definitions

Even(x) ≡ $\exists y\ (x = 2y)$

Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse

Integers

Prove "The sum of two odd numbers is even."

Proof: Let x and y be arbitrary integers.

Suppose that both are odd. Then, we have x = 2a+1 for some integer a and y = 2b+1 for some integer b. Their sum is x+y = (2a+1) + (2b+1) = 2a+2b+2 = 2(a+b+1), so x+y is, by definition, even.

Since x and y were arbitrary, the sum of any two odd integers is even. ■

$$\forall x\ \forall y\ ((\mathrm{Odd}(x) \land \mathrm{Odd}(y)) \rightarrow \mathrm{Even}(x+y))$$

Rational Numbers

Domain of Discourse

Real Numbers

- **A real number x is *rational* iff there exist integers a and b with b≠0 such that x=a/b.**

$\text{Rational}(x) := \exists a\, \exists b\, (((\text{Integer}(a) \wedge \text{Integer}(b)) \wedge (x=a/b)) \wedge b\neq 0)$

Rationality

Domain of Discourse

Real Numbers

Predicate Definitions

Rational(x) := $\exists a \, \exists b \, (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "The product of two rationals is rational."

Formally, prove $\underline{\forall x \, \forall y} \; \underline{((\text{Rational}(x) \land \text{Rational}(y)) \rightarrow \text{Rational}(xy))}$

Let x and y be arbitrary real numbers.

Suppose x and y are rational

Thus xy is rational

Since x and y were arb., we have shown ________

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "The product of two rationals is rational."

Proof: Let x and y be arbitrary reals.

Suppose x and y are rational.

Thus, xy is rational.

Since x and y were arbitrary, we have shown that the product of any two rationals is rational. ■

$\forall x\ \forall y\ ((\text{Rational}(x) \land \text{Rational}(y)) \rightarrow \text{Rational}(xy))$

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

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Rationality

Domain of Discourse

Real Numbers

Predicate Definitions

Rational(x) := $\exists a\ \exists b\ (\boxed{\text{Integer}(a) \land \text{Integer}(b)} \land \boxed{(x = a/b)} \land \boxed{(b \neq 0)})$

Prove: “The product of two rationals is rational.”

Proof: Let x and y be arbitrary rationals.

Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c, d, where d≠0.

$$xy = (a/b)(c/d) = (ac)/(bd)$$

Thus, xy is rational.

Since x and y were arbitrary, we have shown that the product of any two rationals is rational. ■

$\forall x\ \forall y\ ((\text{Rational}(x) \land \text{Rational}(y)) \rightarrow \text{Rational}(xy))$

Rationality

Domain of Discourse

Real Numbers

Predicate Definitions

Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

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Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c, d, where d≠0.

By definition, then, xy is rational.

Since x and y were arbitrary, we have shown that the product of any two rationals is rational. ■

∀x ∀y ((Rational(x) ∧ Rational(y)) → Rational(xy))

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

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Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c, d, where d≠0.

Multiplying, we get that xy = (a/b)(c/d) = (ac)/(bd).

By definition, then, xy is rational.

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Prove: "The product of two rationals is rational."

Proof: Let x and y be arbitrary rationals.

Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c, d, where d≠0.

Multiplying, we get that xy = (a/b)(c/d) = (ac)/(bd).

ac and bd are integers. Also, since b ≠0 and d≠0 we have bd≠0. By definition, then, xy is rational.

Since x and y were arbitrary, we have shown that the product of any two rationals is rational. ■

∀x ∀y ((Rational(x) ∧ Rational(y)) → Rational(xy))

English Proofs

- **High-level language lets us work more quickly**
  - **should not be necessary to spill out every detail**
  - **examples so far**

    skipping Intro ∧ and Elim ∧ (and hence, Commutativity and Associativity)

    skipping Double Negation

    not stating existence claims (immediately apply Elim ∃ to name the object)

    not stating that the implication has been proven (“Suppose X... Thus, Y.” says it already)
  - **(list will grow over time)**

- **English proof is correct if the <u>reader</u> is convinced they could translate it into a formal proof**
  - **the reader is the “compiler” for English proofs**

Proof Strategies

Proof Strategies: Counterexamples

To prove $\neg\forall x\, P(x)$, prove $\overset{\exists x \neg P(x)}{\exists\neg P(x)}$:

- Equivalent by De Morgan's Law
- All we need to do that is find an x where $P(x)$ is **false**
- This example is called a ***counterexample*** to $\forall x\, P(x)$.

e.g. Prove "Not every prime number is odd"

Proof: 2 is a prime that is not odd — a counterexample to the claim that every prime number is odd. ■

An English proof does not need to cite De Morgan's law.

Proof Strategies: Proof by Contrapositive

If we assume $\neg$q and derive $\neg$p, then we have proven $\neg$q → $\neg$p, which is equivalent to proving p → q.

1.1. $\neg q$ Assumption

...

1.3. $\neg p$

1. $\neg q \rightarrow \neg p$ Direct Proof

2. $p \rightarrow q$ Contrapositive: 1

Proof Strategies: Proof by Contrapositive

If we assume $\neg q$ and derive $\neg p$, then we have proven $\neg q \rightarrow \neg p$, which is equivalent to proving $p \rightarrow q$.

We will prove the contrapositive.

Suppose $\neg q$.

...

Thus, $\neg p$.

1.1.	$\neg q$	Assumption
...		
1.3.	$\neg p$	
1.	$\neg q \rightarrow \neg p$	Direct Proof
2.	$p \rightarrow q$	Contrapositive: 1

Proof by Contradiction: One way to prove p

If we assume ¬ p and derive F (a contradiction), then we have proven p.

1.1. $\neg p$ Assumption

...

1.3. F

1. $\neg p \rightarrow$ F Direct Proof
2. $\neg\neg p \vee$ F Law of Implication: 1
3. $p \vee$ F Double Negation: 2
4. p Identity: 3

Proof Strategies: Proof by Contradiction

If we assume $\neg$ p and derive F (a contradiction), then we have proven p.

We will argue by contradiction.

Suppose $\neg p$.

...

This is a contradiction.

1.1.	$\neg p$	Assumption
...		
1.3.	F	
1.	$\neg p \rightarrow$ F	Direct Proof
2.	$\neg\neg p \vee$ F	Law of Implication: 1
3.	$p \vee$ F	Double Negation: 2
4.	p	Identity: 3

Often, we will infer $\neg$R, where R is a prior fact.
Putting these together, we have R $\wedge$ $\neg$R $\equiv$ F

Even and Odd

Predicate Definitions

Even(x) ≡ $\exists y\ (x = 2y)$

Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse

Rationals

Prove: "No integer is both even and odd."

Formally, prove $\neg \exists x\ (\mathrm{Even}(x) \wedge \mathrm{Odd}(x))$

Proof: We will argue by contradiction.

Even and Odd

Predicate Definitions

Even(x) ≡ $\exists y\ (x = 2y)$

Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse

Rationals

Prove: "No integer is both even and odd."

Formally, prove $\neg \exists x\ (\text{Even}(x) \wedge \text{Odd}(x))$

Proof: We will argue by contradiction.

Suppose that x is an integer that is both even and odd.

This is a contradiction. ■

Even and Odd

Predicate Definitions

Even(x) ≡ $\exists y\ (x = 2y)$

Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse

Rationals

Prove: "No integer is both even and odd."

Formally, prove $\neg \exists x\ (\text{Even}(x) \land \text{Odd}(x))$

Proof: We will argue by contradiction.

Suppose that x is an integer that is both even and odd. Then, x=2a for some integer a, and x=2b+1 for some integer b.

This is a contradiction. ■

Even and Odd

Predicate Definitions

Even(x) ≡ $\exists y\ (x = 2y)$

Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse

Rationals

Prove: "No integer is both even and odd."

Formally, prove $\underline{\neg \exists x\ (\text{Even}(x) \land \text{Odd}(x))}$

Proof: We will argue by contradiction.

Suppose that x is an integer that is both even and odd. Then, x=2a for some integer a, and x=2b+1 for some integer b. This means 2a=x=2b+1 and hence 2a-2b=1 and so a-b=½. But a-b is an integer while ½ is not, so they cannot be equal. This is a contradiction. ■

Formally, we've shown Integer(½) ∧ ¬Integer(½) ≡ F.

Proof by Cases

On Homework 3, Task 1 you are asked to show:

- Given $p \rightarrow r$ and $\neg p \rightarrow r$ derive r
- Given $\underline{p \vee q}$, $p \rightarrow r$ and $q \rightarrow r$ derive r

This will mean that…

If we prove p → r and ¬ p → r then we have proven r.

If we prove p ∨ q, p → r and q → r then we have proven r.

Strategies

- Simple proof strategies already do a lot
  - counter examples
  - proof by contrapositive
  - proof by contradiction
  - proof by cases
- Later we will cover a specific strategy that applies to loops and recursion (mathematical induction)

Applications of Predicate Logic

- Remainder of the course will use predicate logic to prove important properties of interesting objects
  - start with math objects that are widely used in CS
  - eventually more CS-specific objects

- Encode domain knowledge in predicate definitions
- Then apply predicate logic to infer useful results

Domain of Discourse

Integers

Predicate Definitions

$\text{Even}(x) \equiv \exists y\ (x = 2 \cdot y)$

$\text{Odd}(x) \equiv \exists y\ (x = 2 \cdot y + 1)$

Number Theory

Number Theory (and applications to computing)

- Branch of Mathematics with direct relevance to computing
- Many significant applications
  - Cryptography & Security
  - Data Structures
  - Distributed Systems
- Important toolkit

Modular Arithmetic

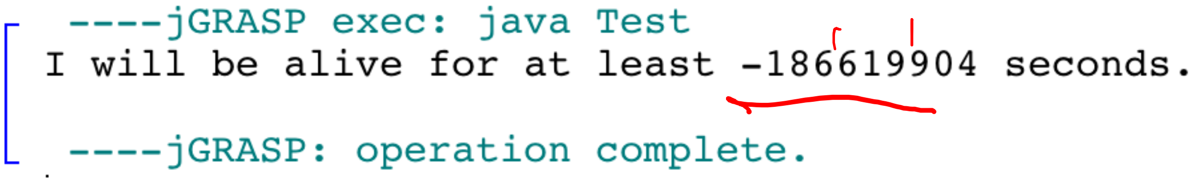
- Arithmetic over a finite domain
- Almost all computation is over a finite domain

I’m ALIVE!

```
                                    365
public class Test {
    final static int SEC_IN_YEAR = ~~364~~*24*60*60*100;
    public static void main(String args[]) {
        System.out.println(
            “I will be alive for at least ” +
            SEC_IN_YEAR ~~* 101~~ + “ seconds.”
        );
    }
}
```

I’m ALIVE!

```
public class Test {
    final static int SEC_IN_YEAR = 364*24*60*60*100;
    public static void main(String args[]) {
        System.out.println(
            “I will be alive for at least ” +
            SEC_IN_YEAR * 101 + “ seconds.”
        );
    }
}
```



```
 ----jGRASP exec: java Test
I will be alive for at least -186619904 seconds.

 ----jGRASP: operation complete.
```

Divisibility

Domain of Discourse
Integers

Definition: “b divides a”

For a, b with $b \neq 0$:

$$b \mid a \leftrightarrow \exists q \, (a = qb)$$

Check Your Understanding. Which of the following are true?

$5 \mid 1$ ✗ $25 \mid 5$ ✗ $5 \mid 0$ ✓ $3 \mid 2$ ✗

$1 \mid 5$ ✓ $5 \mid 25$ ✓ ~~$0 \mid 5$~~ $2 \mid 3$ ✗

$5 \cdot 5 = 25$

Divisibility

Domain of Discourse
Integers

Definition: “b divides a”

For a, b with $b \neq 0$:

$$b \mid a \leftrightarrow \exists q \; (a = qb)$$

Check Your Understanding. Which of the following are true?

5 | 1

5 | 1 iff 1 = 5k

25 | 5

25 | 5 iff 5 = 25k

[✓ 5 | 0]

5 | 0 iff 0 = 5k

3 | 2

3 | 2 iff 2 = 3k

[✓ 1 | 5]

1 | 5 iff 5 = 1k

[✓ 5 | 25]

5 | 25 iff 25 = 5k

0 | 5

0 | 5 iff 5 = 0k

2 | 3

2 | 3 iff 3 = 2k

Recall: Elementary School Division

For a, b with $b > 0$, we can divide b into a.

If $b \mid a$, then, by definition, we have $a = qb$ for some q.
The number q is called the ***quotient***.

Dividing both sides by b, we can write this as

$$\frac{a}{b} = q$$

(We want to stick to integers, though, so we'll write $a = qb$.)

Recall: Elementary School Division

For a, b with $b > 0$, we can divide b into a.

If $b \nmid a$, then we end up with a *remainder* r with $0 < r < b$.
Now,

instead of $\frac{a}{b} = q$ we have $\underline{\frac{a}{b} = q + \frac{r}{b}}$

Multiplying both sides by b gives us $a = qb + r$
(A bit nicer since it has no fractions.)

Recall: Elementary School Division

For a, b with $b > 0$, we can divide b into a.

If $b \mid a$, then we have $a = qb$ for some q.
If $b \nmid a$, then we have $a = qb + r$ for some q, r with $0 < r < b$.

In general, we have $a = qb + r$ for some q, r with $0 \leq r < b$, where $r = 0$ iff $b \mid a$.

Division Theorem

Domain of Discourse
Integers

Division Theorem

For a, b with $b > 0$
there exist *unique* integers q, r with $0 \leq r < b$
such that $a = qb + r$.

To put it another way, if we divide *b* into *a*, we get a unique quotient $q = a\ \mathbf{div}\ b$
and non-negative remainder $r = a\ \mathbf{mod}\ b$

a/b

$a\%b$

Note: r ≥ 0 even if a < 0.
Not quite the same as **`a % b`**.

Division Theorem

Domain of Discourse

Integers

Division Theorem

For a, b with $b > 0$

there exist *unique* integers *q*, *r* with $0 \leq r < b$

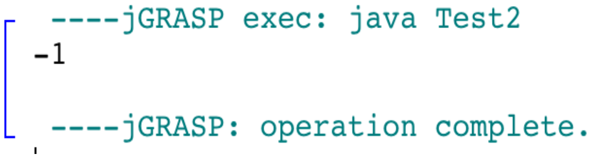
such that $a = qb + r$.

To put it another way, if we divide *b* into *a*, we get a unique quotient ***q* = *a* div *b***

and non-negative remainder ***r* = *a* mod *b*** %

```
public class Test2 {
    public static void main(String args[]) {
        int a = -5;
        int b = 2;
        System.out.println(a % b);
    }
}
```

$-5/2 = -2$



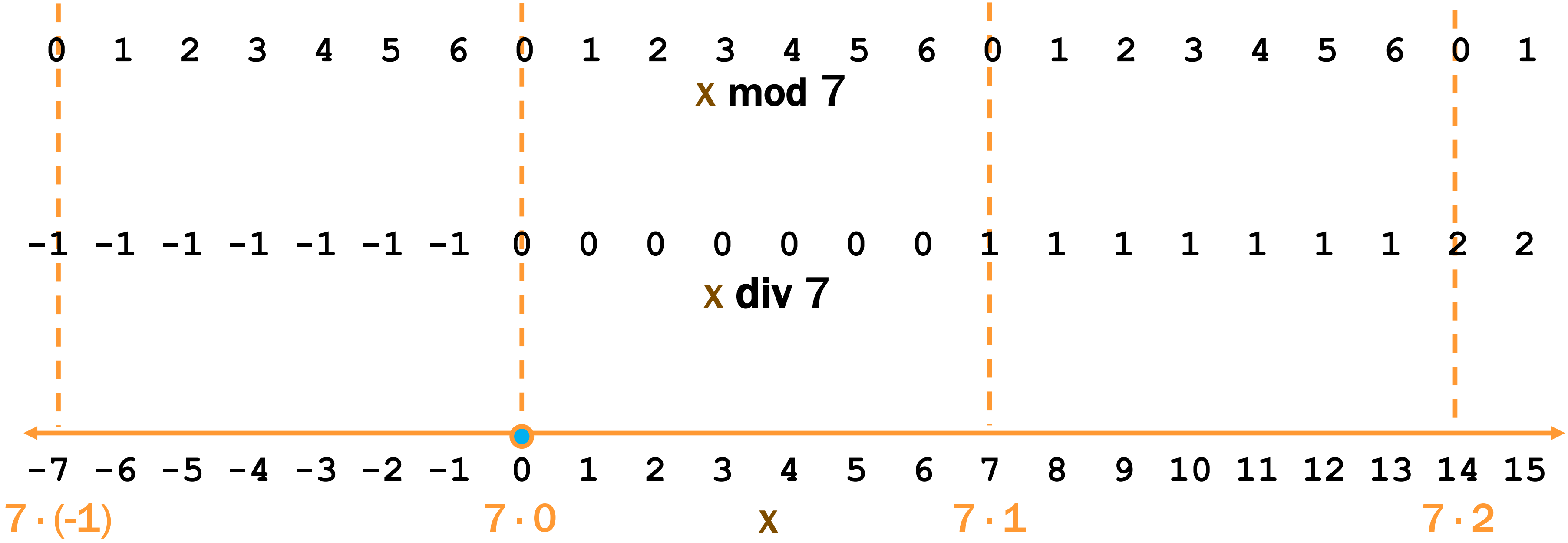
```
 ----jGRASP exec: java Test2
-1

 ----jGRASP: operation complete.
```

Note: r ≥ 0 even if a < 0.
Not quite the same as **`a % b`**.

div and mod

$x = 7 \cdot (x \text{ div } 7) + (x \text{ mod } 7)$



x mod 7	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1
x div 7	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	2	2
x	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15