Lecture 14: Induction

Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily

- **Particularly useful for reasoning about programs!**
  ```
  for (int i=0; i < n; n++) { … }
  ```
  - Show P(i) holds after i times through the loop

Prove $\forall a, b, m > 0 \; \forall \, k \in \mathbb{N} \; ((a \equiv b \; (mod \; m)) \to (a^k \equiv b^k \; (mod \; m))$

Let $a, b, m > 0$ be arbitrary. Let $k \in \mathbb{N}$ be arbitrary.

Suppose that $a \equiv b \; (\text{mod} \; m)$.

We know that by multiplying congruences we get

$$\left(a \equiv b \; (\text{mod} \; m) \;\wedge\; a \equiv b \; (\text{mod} \; m) \right) \to a^2 \equiv b^2 \; (\text{mod} \; m)$$

Then, repeating this many times, we have:

$$\left(a^2 \equiv b^2 \; (\text{mod} \; m) \;\wedge\; a \equiv b \; (\text{mod} \; m) \right) \to a^3 \equiv b^3 \; (\text{mod} \; m)$$

$$\left(a^3 \equiv b^3 \; (\text{mod} \; m) \;\wedge\; a \equiv b \; (\text{mod} \; m) \right) \to a^4 \equiv b^4 \; (\text{mod} \; m)$$

...

$$\left(a^{k-1} \equiv b^{k-1} \; (\text{mod} \; m) \;\wedge\; a \equiv b \; (\text{mod} \; m) \right) \to a^k \equiv b^k \; (\text{mod} \; m)$$

The "..." is a problem! We don't have a proof rule that allows us to say "do this over and over".

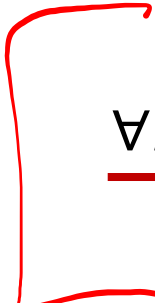# But there such a property of the natural numbers!

Domain: Natural Numbers

$$\textcircled{1}\, P(0)$$

$$\textcircled{2}\, \forall k\ (P(k) \longrightarrow P(k+1))$$

$$\therefore \forall n\ P(n)$$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$P(0)$$
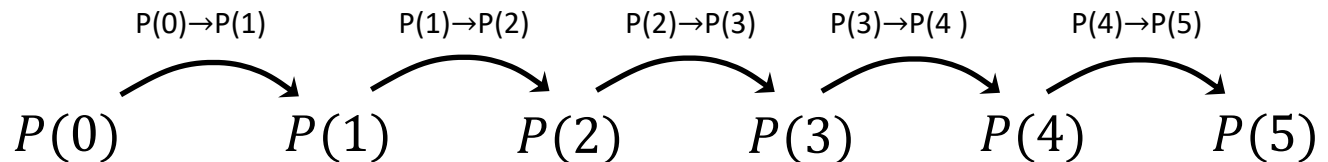$$\forall k \ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n \ P(n)$$

How do the givens prove P(5)?

Induction Is A Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall k \ (P(k) \longrightarrow P(k+1))$$
$$\therefore \ \forall n \ P(n)$$

How do the givens prove P(5)?

P(0)→P(1)   P(1)→P(2)   P(2)→P(3)   P(3)→P(4)   P(4)→P(5)

$P(0)$ $P(1)$ $P(2)$ $P(3)$ $P(4)$ $P(5)$

First, we have P(0).

Since P(n) $\rightarrow$ P(n+1) for all n, we have P(0) $\rightarrow$ P(1).

    Since P(0) is true and P(0) $\rightarrow$ P(1), by Modus Ponens, P(1) is true.

Since P(n) $\rightarrow$ P(n+1) for all n, we have P(1) $\rightarrow$ P(2).

    Since P(1) is true and P(1) $\rightarrow$ P(2), by Modus Ponens, P(2) is true.

…

Using The Induction Rule In A Formal Proof

$$P(0)$$
$$\forall k \ (P(k) \longrightarrow P(k+1))$$

$$\therefore \forall n \ P(n)$$

Using The Induction Rule In A Formal Proof

$$① \quad P(0)$$

$$② \quad \frac{\forall k \ (P(k) \longrightarrow P(k+1))}{\therefore \ \forall n \ P(n)}$$

1. P(0)

4. $\forall$k (P(k) $\rightarrow$ P(k+1))
5. $\forall$n P(n)   Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$P(0)$$
$$\forall k \ (P(k) \longrightarrow P(k + 1))$$
$$\therefore \ \forall n \ P(n)$$

1. P(0)
2. Let k be an arbitrary integer ≥ 0

3. P(k) $\rightarrow$ P(k+1)
4. $\forall$k (P(k) $\rightarrow$ P(k+1))   Intro $\forall$: 2, 3
5. $\forall$n P(n)   Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$P(0)$$
$$\forall k \ (P(k) \longrightarrow P(k+1))$$
$$\overline{}$$
$$\therefore \ \forall n \ P(n)$$

$\forall x,$

$x \geq 0$

$\rightarrow P(x)$

1. P(0)
2. Let k be an arbitrary integer ≥ 0
    3.1. P(k)   Assumption
    3.2. …
    3.3. P(k+1)
3.   P(k) $\rightarrow$ P(k+1)   Direct Proof Rule
4. $\forall$k (P(k) $\rightarrow$ P(k+1))   Intro $\forall$: 2, 3
5. $\forall$n P(n)   Induction: 1, 4

Translating to an English Proof

$$P(0)$$
$$\forall k \ (P(k) \longrightarrow P(k+1))$$
$$\therefore \ \forall n \ P(n)$$

1. Prove P(0) — **Base Case**

2. Let k be an arbitrary integer ≥ 0 — **Inductive Hypothesis**

    3.1. Suppose that P(k) is true

    3.2. …

    3.3. Prove P(k+1) is true — **Inductive Step**

3. $P(k) \rightarrow P(k+1)$ Direct Proof Rule

4. $\forall k \ (P(k) \rightarrow P(k+1))$ Intro $\forall$: 2, 3

5. $\forall n \ P(n)$ Induction: 1, 4

Conclusion

Translating to an English Proof

1. Prove P(0)	**Base Case**
2. Let k be an arbitrary integer ≥ 0	**Inductive Hypothesis**
3.1. Assume that P(k) is true	
3.2. …	**Inductive Step**
3.3. Prove P(k+1) is true	
3. P(k) → P(k+1) Direct Proof Rule	
4. ∀k (P(k) → P(k+1)) Intro ∀: 2, 3	
5. ∀n P(n) Induction: 1, 4	**Conclusion**

Induction English Proof Template

[…Define P(n)…]
We will show that $P(n)$ is true for every $n \in \mathbb{N}$ by Induction.
Base Case: *[…proof of $P(0)$ here…]*
Induction Hypothesis:
    Suppose that $P(k)$ is true for an arbitrary $k \in \mathbb{N}$.
Induction Step:
    […proof of $P(k+1)$ here…]
    *The proof of $P(k+1)$ **must** invoke the IH somewhere.*
So, the claim is true by induction.

Inductive Proofs In 5 Easy Steps

Proof:

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."

2. "Base Case:" Prove $P(0)$

3. "Inductive Hypothesis:

   Suppose $P(k)$ is true for an arbitrary integer $k \geq 0$"

4. "Inductive Step:" Prove that $P(k+1)$ is true.

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)

5. "Conclusion: Result follows by induction"

What is $1 + 2 + 4 + \ldots + 2^n$?

- 1 $\qquad = \quad 1$
- $1 + 2$ $\qquad = \quad 3$
- $1 + 2 + 4$ $\qquad = \quad 7$
- $1 + 2 + 4 + 8$ $\qquad = \quad 15$
- $1 + 2 + 4 + 8 + 16$ $\qquad = \quad 31$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

   We could prove it for $n = 1, n = 2, n = 3, \ldots$ but that would literally take forever.

   Good that we have induction!

Prove $1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** $P(n)$ **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** $P(n)$ **is true for all natural numbers by induction.**

Prove $1 + 2 + 4 + ... + 2^n = 2^{n+1} - 1$

1. **Let** $P(n)$ **be** "$2^0 + 2^1 + ... + 2^n = 2^{n+1} - 1$". **We will show** $P(n)$ **is true for all natural numbers by induction.**

2. **Base Case** $(n=0)$: $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** $P(0)$ **is true.**

Prove $1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$

1. **Let** $P(n)$ **be** "$2^0 + 2^1 + \dots + 2^n = 2^{n+1} - 1$". **We will show** $P(n)$ **is true for all natural numbers by induction.**

2. **Base Case** ($n=0$): $\quad 2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** $P(0)$ **is true.**

3. **Induction Hypothesis: Suppose that** $P(k)$ **is true for some arbitrary integer** $k \geq 0$, **i.e., that** $\boxed{2^0 + 2^1 + \dots + 2^k = 2^{k+1} - 1.}$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. Let P(n) be "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". We will show P(n) is true for all natural numbers by induction.

2. Base Case (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ so P(0) is true.

3. Induction Hypothesis: Suppose that P(k) is true for some arbitrary integer k ≥ 0, i.e., that $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.

4. Induction Step:

   Goal: Show P(k+1), i.e. show $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. Let $P(n)$ be "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". We will show $P(n)$ is true for all natural numbers by induction.

2. Base Case ($n=0$): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ so $P(0)$ is true.

3. Induction Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 0$, i.e., that $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.

4. Induction Step:

   $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$ by IH

   Adding 2^{k+1} to both sides, we get:

   $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

   So, we have $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly $P(k+1)$.

Prove $1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$

1. Let $P(n)$ be "$2^0 + 2^1 + \dots + 2^n = 2^{n+1} - 1$". We will show $P(n)$ is true for all natural numbers by induction.

2. Base Case ($n=0$): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ so $P(0)$ is true.

3. Induction Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 0$, i.e., that $2^0 + 2^1 + \dots + 2^k = 2^{k+1} - 1$.

4. Induction Step:

   We can calculate

   $$2^0 + 2^1 + \dots + 2^k + 2^{k+1} = (2^0 + 2^1 + \dots + 2^k) + 2^{k+1}$$
   $$= (2^{k+1} - 1) + 2^{k+1} \qquad \text{by the IH}$$
   $$= 2(2^{k+1}) - 1$$
   $$= 2^{k+2} - 1,$$

   which is exactly $P(k+1)$.

   Alternative way of writing the inductive step

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. Let $P(n)$ be "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". We will show $P(n)$ is true for all natural numbers by induction.

2. Base Case ($n=0$): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ so $P(0)$ is true.

3. Induction Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 0$, i.e., that $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.

4. Induction Step:

   We can calculate

   $$2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = (2^0 + 2^1 + \ldots + 2^k) + 2^{k+1}$$
   $$= (2^{k+1} - 1) + 2^{k+1} \qquad \text{by the IH}$$
   $$= 2(2^{k+1}) - 1$$
   $$= 2^{k+2} - 1,$$

   which is exactly P(k+1).

5. Thus $P(n)$ is true for all $n \in \mathbb{N}$, by induction.

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + ... + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**

Summation Notation

$$\sum_{i=0}^{n} i = 0 + 1 + 2 + 3 + \ldots + n$$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + ... + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**

2. **Base Case** (n=0):   0 = 0(0+1)/2. **Therefore** P(0) **is true.**

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2. **Base Case** (n=0):   0 = 0(0+1)/2. **Therefore** P(0) **is true.**

3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0. **I.e., suppose** 1 + 2 + …+ k = k(k+1)/2

"some" or "an"
not <u>any</u>!

Prove $1 + 2 + 3 + \dots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + ... + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**

2. **Base Case** (n=0):   0 = 0(0+1)/2. **Therefore** P(0) **is true.**

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Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + ... + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**

2. **Base Case** (n=0):   0 = 0(0+1)/2. **Therefore** P(0) **is true.**

3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0. **I.e., suppose** 1 + 2 + ...+ k = k(k+1)/2

4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** 1 + 2 + ...+ k+ (k+1) = (k+1)(k+2)/2

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**

2. **Base Case** (n=0):   0 = 0(0+1)/2. **Therefore** P(0) **is true.**

3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0. **I.e., suppose** 1 + 2 + …+ k = k(k+1)/2

4. **Induction Step:**

   1 + 2 + … + k + (k+1) = (1 + 2 + … + k) + (k+1)

   = k(k+1)/2 + (k+1) **by IH**

   = (k+1)(k/2 + 1)

   = (k+1)(k+2)/2

   $k/2 + 1 = \dfrac{k+2}{2}$

   So, we have shown 1 + 2 + … + k + (k+1) = (k+1)(k+2)/2,

   which is exactly P(k+1).

5. **Thus** P(n) **is true for all** n ∈ ℕ, **by induction.**

Induction: Changing the start line

- **What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b?**

- **Define predicate $Q(k) = P(k + b)$ for all k.**
  - **Then $\forall n \; Q(n) \equiv \forall n \geq b \;\; P(n)$**

- **Ordinary induction for Q:**
  - **Prove $Q(0) \equiv P(b)$**
  - **Prove**
    $$\forall k \left(Q(k) \longrightarrow Q(k+1) \right) \equiv \forall k \geq b \left(P(k) \longrightarrow P(k+1) \right)$$

Inductive Proofs starting at b in 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."

2. "Base Case:" Prove $P(b)$

3. "Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary integer $k \geq b$"

4. "Inductive Step:" Prove that $P(k + 1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k + 1)$!!)

5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Prove $3^n \geq n^2 + 3$ **for all** $n \geq 2$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let P(n) be "$3^n \geq n^2 + 3$". We will show P(n) is true for all integers $n \geq 2$ by induction.

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2 + 3$". **We will show** P(n) i**s true for all integers** n ≥ 2 **by induction.**

2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) i**s true for all integers** $n \geq 2$ **by induction.**

2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**

3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$. **I.e., suppose** $3^k \geq k^2+3$.

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let P(n) be "$3^n \geq n^2+3$". We will show P(n) is true for all integers $n \geq 2$ by induction.

2. Base Case (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ so P(2) is true.

3. Inductive Hypothesis: Suppose that P(k) is true for some arbitrary integer $k \geq 2$. I.e., suppose $3^k \geq k^2+3$.

4. Inductive Step:

> Goal: Show P(k+1), i.e. show $3^{k+1} \geq (k+1)^2+3$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let P(n) be "$3^n \geq n^2+3$". We will show P(n) is true for all integers $n \geq 2$ by induction.

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3. Inductive Hypothesis: Suppose that P(k) is true for some arbitrary integer $k \geq 2$. I.e., suppose $3^k \geq k^2+3$.

4. Inductive Step:

   Goal: Show P(k+1), i.e. show $3^{k+1} \geq (k+1)^2+3 = k^2+2k+4$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) i**s true for all integers** n $\geq$ 2 **by induction.**

2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**

3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k $\geq$ 2**. I.e., suppose** $3^k \geq k^2+3$.

4. **Inductive Step:**

   Goal: Show P(k+1)**, i.e. show** $3^{k+1} \geq (k+1)^2+3 = k^2+2k+4$

   $3^{k+1} = 3(3^k)$

   $\geq 3(k^2+3)$ **by the IH**

   $= 3k^2+9$

   $= k^2+2k^2+9$

   $\geq k^2+2k+4 = (k+1)^2+3$ **since** k $\geq$ 2.

   Therefore P(k+1) **is true.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** n ≥ 2 **by induction.**

2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**

3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 2. **I.e., suppose** $3^k \geq k^2+3$.

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   Goal: Show P(k+1)**, i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

   $3^{k+1} = 3(3^k)$
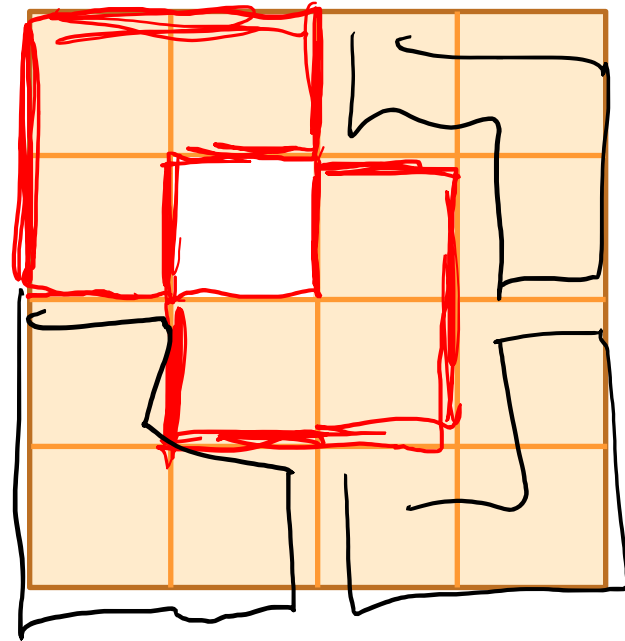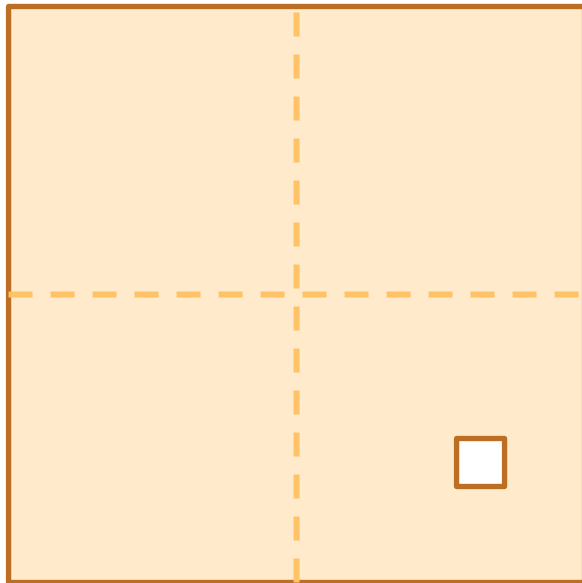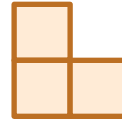
   $\geq 3(k^2+3)$ **by the IH**

   $= k^2+2k^2+9$

   $\geq k^2+2k+4 = (k+1)^2+3$ **since** k ≥ 2.

   Therefore P(k+1) **is true.**

5. **Thus** P(n) **is true for all integers** n ≥ 2, **by induction.**

Checkerboard Tiling

- **Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:**

Checkerboard Tiling

1. Let $P(n)$ be any $2^n \times 2^n$ checkerboard with one square removed can be tiled with .
    We prove $P(n)$ for all $n \geq 1$ by induction on n.

Checkerboard Tiling

1. **Let** P(n) **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with** **.**
    We prove P(n) **for all** $n \geq 1$ **by induction on** n.

2. **Base Case:** n=1

Checkerboard Tiling

1. Let $P(n)$ be any $2^n \times 2^n$ checkerboard with one square removed can be tiled with .
    We prove $P(n)$ for all $n \geq 1$ by induction on n.

2. Base Case: $n=1$

3. Inductive Hypothesis: Assume $P(k)$ for some arbitrary integer $k \geq 1$

Checkerboard Tiling

1. **Let** $P(n)$ **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with** **.**
    We prove $P(n)$ **for all** $n \geq 1$ **by induction on** n.

2. **Base Case:** $n=1$

3. **Inductive Hypothesis:** **Assume** $P(k)$ **for some arbitrary integer** $k \geq 1$

4. **Inductive Step: Prove** $P(k+1)$
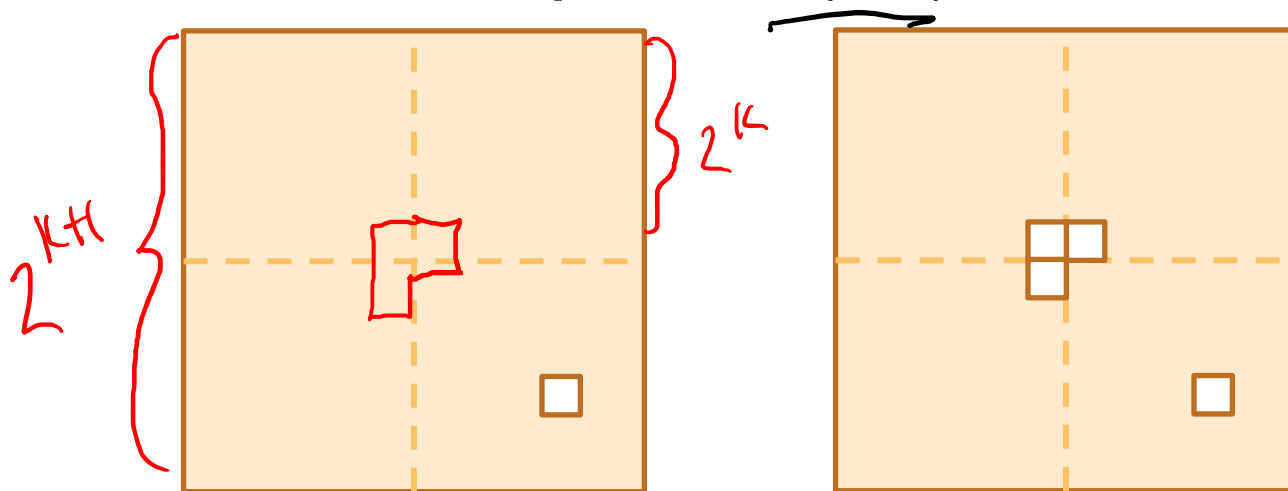
Apply IH to each quadrant then fill with extra tile.