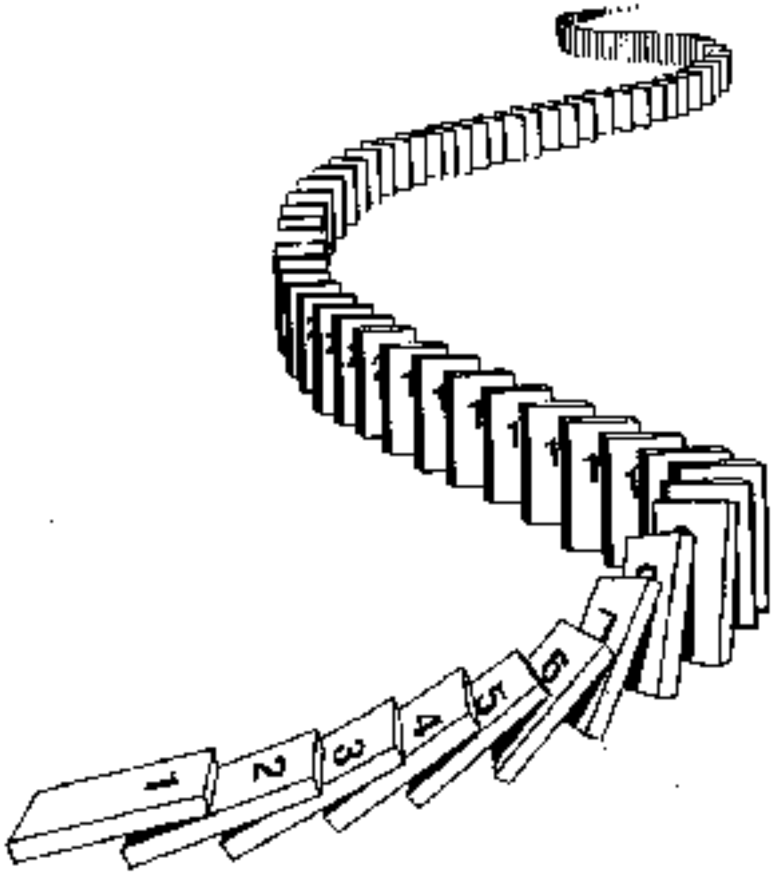
CSE 311: Foundations of Computing

Lecture 14: Induction



Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily

- **Particularly useful for reasoning about programs!**

  ```
  for (int i=0; i < n; n++) { … }
  ```

  - Show P(i) holds after i times through the loop

Prove $\forall a, b, m > 0\ \forall\ k \in \mathbb{N}\ ((a \equiv b\ (mod\ m)) \rightarrow (a^k \equiv b^k\ (mod\ m))$

Let $a, b, m > 0$ be arbitrary. Let $k \in \mathbb{N}$ be arbitrary.

Suppose that $a \equiv b \pmod{m}$.

We know that by multiplying congruences we get

$$\left(a \equiv b \pmod{m}\ \land\ a \equiv b \pmod{m}\right) \rightarrow a^2 \equiv b^2 \pmod{m}$$

Then, repeating this many times, we have:

$$\left(a^2 \equiv b^2 \pmod{m}\ \land\ a \equiv b \pmod{m}\right) \rightarrow a^3 \equiv b^3 \pmod{m}$$
$$\left(a^3 \equiv b^3 \pmod{m}\ \land\ a \equiv b \pmod{m}\right) \rightarrow a^4 \equiv b^4 \pmod{m}$$

…

$$\left(a^{k-1} \equiv b^{k-1} \pmod{m}\ \land\ a \equiv b \pmod{m}\right) \rightarrow a^k \equiv b^k \pmod{m}$$

The "…" is a problem! We don't have a proof rule that allows us to say "do this over and over".

But there such a property of the natural numbers!

Domain: Natural Numbers

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$\begin{tikzcd} P(0) \arrow[r, bend left, "P(0)\rightarrow P(1)"] & P(1) \arrow[r, bend left, "P(1)\rightarrow P(2)"] & P(2) \arrow[r, bend left, "P(2)\rightarrow P(3)"] & P(3) \arrow[r, bend left, "P(3)\rightarrow P(4)"] & P(4) \arrow[r, bend left, "P(4)\rightarrow P(5)"] & P(5) \end{tikzcd}$$

First, we have P(0).
Since P(n) → P(n+1) for all n, we have P(0) → P(1).
    Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.
Since P(n) → P(n+1) for all n, we have P(1) → P(2).
    Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.
...

Using The Induction Rule In A Formal Proof

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. P(0)

4. $\forall$k (P(k) $\rightarrow$ P(k+1))
5. $\forall$n P(n) Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. $P(0)$
2. Let k be an arbitrary integer ≥ 0

3. $P(k) \rightarrow P(k+1)$
4. $\forall k\ (P(k) \rightarrow P(k+1))$ Intro $\forall$: 2, 3
5. $\forall n\ P(n)$ Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{P(0) \qquad \forall k\ (P(k) \longrightarrow P(k+1))}{\therefore \forall n\ \ P(n)}$$

1. P(0)
2. Let k be an arbitrary integer ≥ 0
    - 3.1. P(k) Assumption
    - 3.2. ...
    - 3.3. P(k+1)
3. P(k) → P(k+1) Direct Proof Rule
4. ∀k (P(k) → P(k+1)) Intro ∀: 2, 3
5. ∀n P(n) Induction: 1, 4

Translating to an English Proof

$$\frac{P(0) \qquad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)

Base Case

2. Let k be an arbitrary integer ≥ 0
    3.1. Suppose that P(k) is true

Inductive Hypothesis

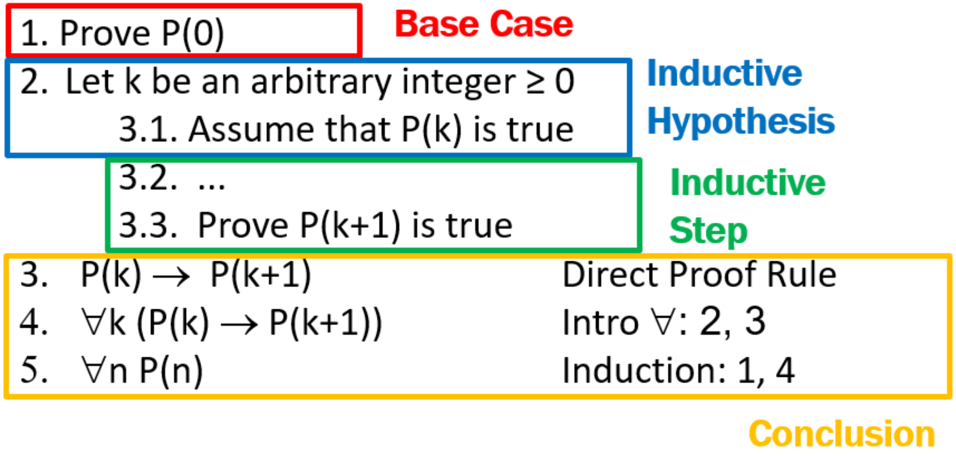
    3.2. ...
    3.3. Prove P(k+1) is true

Inductive Step

3. P(k) → P(k+1) Direct Proof Rule
4. ∀k (P(k) → P(k+1)) Intro ∀: 2, 3
5. ∀n P(n) Induction: 1, 4

Conclusion

Translating to an English Proof



Base Case

1. Prove P(0)

Inductive Hypothesis

2. Let k be an arbitrary integer ≥ 0
    3.1. Assume that P(k) is true

Inductive Step

    3.2. ...
    3.3. Prove P(k+1) is true

Conclusion

3. P(k) → P(k+1) — Direct Proof Rule
4. ∀k (P(k) → P(k+1)) — Intro ∀: 2, 3
5. ∀n P(n) — Induction: 1, 4

Induction English Proof Template

[...Define P(n)...]

We will show that $P(n)$ is true for every $n \in \mathbb{N}$ by Induction.

Base Case: *[...proof of $P(0)$ here...]*

Induction Hypothesis:

Suppose that $P(k)$ is true for an arbitrary $k \in \mathbb{N}$.

Induction Step:

[...proof of $P(k+1)$ here...]

*The proof of $P(k+1)$ **must** invoke the IH somewhere.*

So, the claim is true by induction.

Inductive Proofs In 5 Easy Steps

Proof:

1. “Let $P(n)$ be… . We will show that $P(n)$ is true for every $n \geq 0$ by Induction.”
2. “Base Case:” Prove $P(0)$
3. “Inductive Hypothesis:

   Suppose $P(k)$ is true for an arbitrary integer $k \geq 0$”
4. “Inductive Step:” Prove that $P(k+1)$ is true.

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don’t assume $P(k+1)$!!)
5. “Conclusion: Result follows by induction”

What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 4 = 7$
- $1 + 2 + 4 + 8 = 15$
- $1 + 2 + 4 + 8 + 16 = 31$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

> **We could prove it for $n = 1, n = 2, n = 3, \ldots$ but that would literally take forever.**
>
> **Good that we have induction!**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

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1. **Let** P(n) **be** “$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$”. **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** “$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$”. **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** $P(n)$ **be** “$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$”. **We will show** $P(n)$ **is true for all natural numbers by induction.**
2. **Base Case** $(n=0)$: $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** $P(0)$ **is true.**
3. **Induction Hypothesis: Suppose that** $P(k)$ **is true for some arbitrary integer** $k \geq 0$, **i.e., that** $\boxed{2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1.}$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$**.**
4. **Induction Step:**

   Goal: Show P(k+1)**, i.e. show** $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** “$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$”. **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$, **i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. **Induction Step:**

$$2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1 \quad \textbf{by IH}$$

Adding 2^{k+1} **to both sides, we get:**

$$2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

So, we have $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, **which is exactly** P(k+1).

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

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1. **Let** P(n) **be** “$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$”. **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$, **i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. **Induction Step:**

   We can calculate

$$
\begin{aligned}
2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0+2^1+ \ldots + 2^k) + 2^{k+1} \\
&= (2^{k+1} - 1) + 2^{k+1} && \textbf{by the IH} \\
&= 2(2^{k+1}) - 1 \\
&= 2^{k+2} - 1,
\end{aligned}
$$

   which is exactly P(k+1).

Alternative way of writing the inductive step

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$, **i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. **Induction Step:**

   We can calculate

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0+2^1+ \ldots + 2^k) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} && \textbf{by the IH} \\ &= 2(2^{k+1}) - 1 \\ &= 2^{k+2} - 1, \end{aligned}$$

   which is exactly P(k+1).

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$, **by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

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1. **Let** P(n) **be** “0 + 1 + 2 + … + n = n(n+1)/2”. **We will show** P(n) **is true for all natural numbers by induction.**

Summation Notation

$$\sum_{i=0}^{n} i = 0 + 1 + 2 + 3 + \ldots + n$$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** “0 + 1 + 2 + … + n = n(n+1)/2”. **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**

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2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**. I.e., suppose** 1 + 2 + …+ k = k(k+1)/2

“some” or “an”
not <u>any</u>!

Prove $1 + 2 + 3 + \dots + n = n(n+1)/2$

1. **Let** P(n) **be** “0 + 1 + 2 + … + n = n(n+1)/2”. **We will show** P(n) **is true for all natural numbers by induction.**
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Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

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2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0. **I.e., suppose** 1 + 2 + …+ k = k(k+1)/2
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** 1 + 2 + …+ k+ (k+1) = (k+1)(k+2)/2

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** “$0 + 1 + 2 + \ldots + n = n(n+1)/2$”. **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $0 = 0(0+1)/2$. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**. I.e., suppose** $1 + 2 + \ldots + k = k(k+1)/2$
4. **Induction Step:**

$$\begin{aligned} 1 + 2 + \ldots + k + (k+1) &= (1 + 2 + \ldots + k) + (k+1) \\ &= k(k+1)/2 + (k+1) \quad \textbf{by IH} \\ &= (k+1)(k/2 + 1) \\ &= (k+1)(k+2)/2 \end{aligned}$$

   So, we have shown $1 + 2 + \ldots + k + (k+1) = (k+1)(k+2)/2$,

   which is exactly P(k+1).

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**

Induction: Changing the start line

- What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b?

- Define predicate $Q(k) = P(k+b)$ for all k.
  - Then $\forall n\, Q(n) \equiv \forall n \geq b\ \ P(n)$

- Ordinary induction for Q:
  - Prove $Q(0) \equiv P(b)$
  - Prove

    $\forall k\ \big(Q(k) \longrightarrow Q(k+1)\big) \equiv \forall k \geq b\big(P(k) \longrightarrow P(k+1)\big)$

Inductive Proofs starting at $\boldsymbol{b}$ in 5 Easy Steps

1. “Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq \boldsymbol{b}$ by induction.”
2. “Base Case:” Prove $P(\boldsymbol{b})$
3. “Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary integer $k \geq \boldsymbol{b}$”
4. “Inductive Step:” Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don’t assume $P(k+1)$!!)
5. “Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$”

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$. **I.e., suppose** $3^k \geq k^2+3$.

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** “$3^n \geq n^2+3$”. **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
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4. **Inductive Step:**

   Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3 = k^2+2k+4$

Prove $3^n \geq n^2 + 3$ **for all** $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$. **I.e., suppose** $3^k \geq k^2+3$.
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

$$\begin{aligned} 3^{k+1} &= 3(3^k) \\ &\geq 3(k^2+3) \text{ by the IH} \\ &= 3k^2+9 \\ &= k^2+2k^2+9 \\ &\geq k^2+2k+4 = (k+1)^2+3 \text{ since } k \geq 2. \end{aligned}$$

Therefore P(k+1) **is true.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** $P(n)$ **be** "$3^n \geq n^2+3$". **We will show** $P(n)$ **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** $(n=2)$: $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** $P(2)$ **is true.**
3. **Inductive Hypothesis: Suppose that** $P(k)$ **is true for some arbitrary integer** $k \geq 2$**. I.e., suppose** $3^k \geq k^2+3$**.**
4. **Inductive Step:**

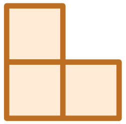
Goal: Show $P(k+1)$**, i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

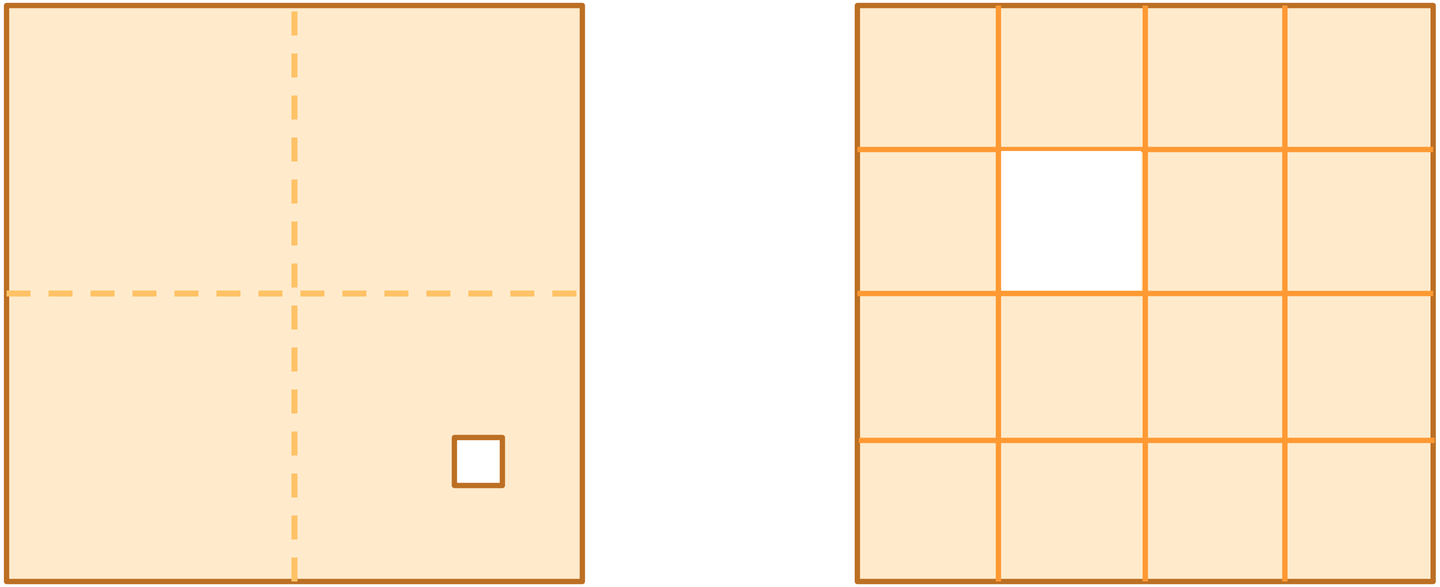
$$\begin{aligned} 3^{k+1} &= 3(3^k) \\ &\geq 3(k^2+3) \text{ by the IH} \\ &= k^2+2k^2+9 \\ &\geq k^2+2k+4 = (k+1)^2+3 \text{ since } k \geq 2. \end{aligned}$$

Therefore $P(k+1)$ **is true.**

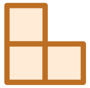
5. **Thus** $P(n)$ **is true for all integers** $n \geq 2$**, by induction.**

Checkerboard Tiling

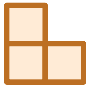
- Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 



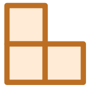
Checkerboard Tiling

1. **Let $P(n)$ be any $2^n \times 2^n$ checkerboard with one square removed can be tiled with  .**
   We prove $P(n)$ for all $n \geq 1$ by induction on n.

Checkerboard Tiling

1. **Let** $P(n)$ **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove $P(n)$ **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** $n=1$

Checkerboard Tiling

1. **Let** $P(n)$ **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove $P(n)$ **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** $n=1$ 
3. **Inductive Hypothesis: Assume** $P(k)$ **for some arbitrary integer** $k \geq 1$

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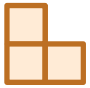
1. **Let $P(n)$ be any $2^n \times 2^n$ checkerboard with one square removed can be tiled with**  **.**
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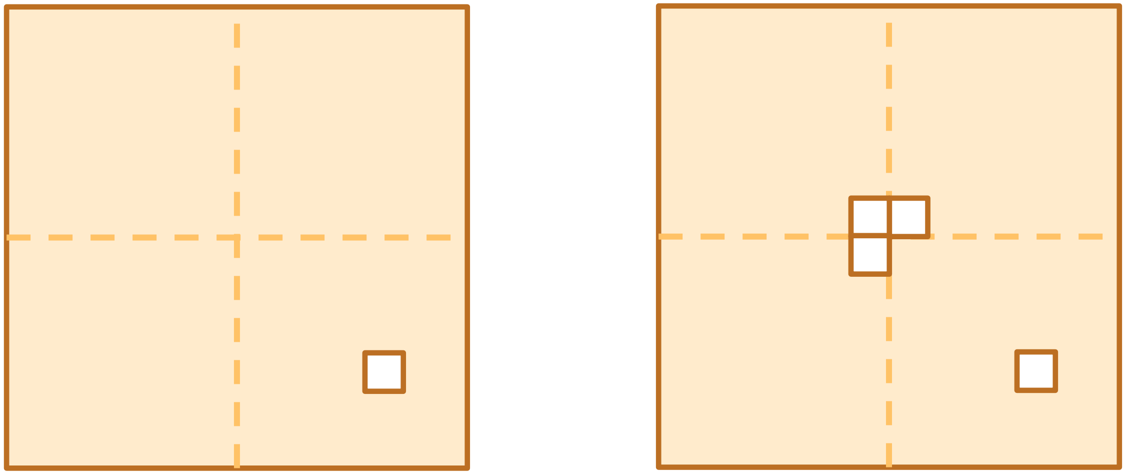
4. **Inductive Step: Prove $P(k+1)$**



Apply IH to each quadrant then fill with extra tile.