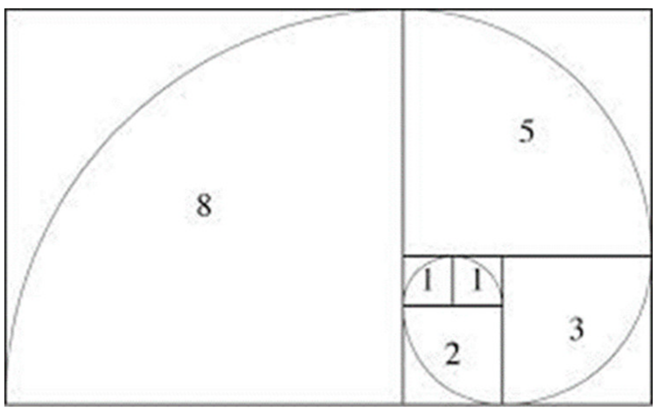
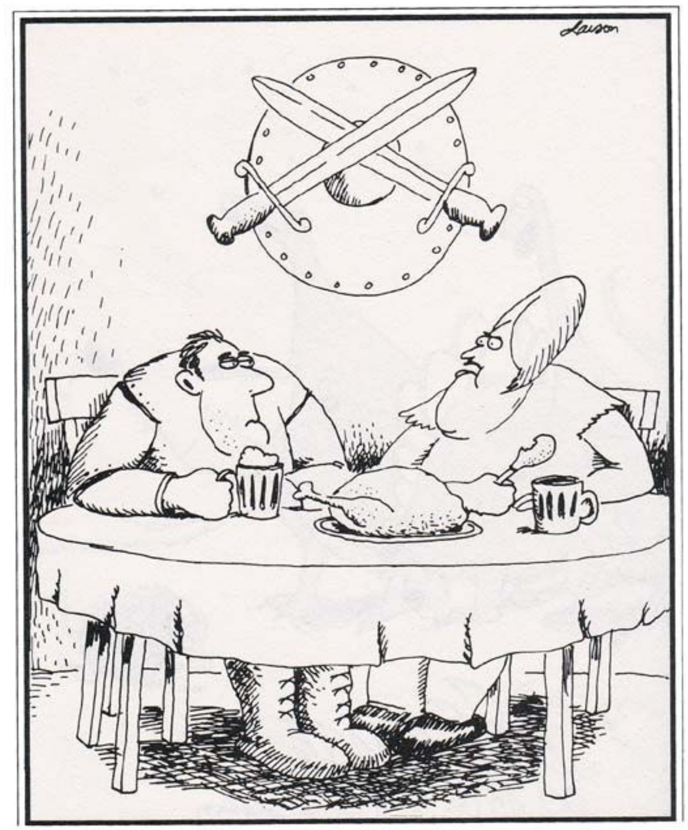
CSE 311: Foundations of Computing

Lecture 15: Recursion & Strong Induction Applications: Fibonacci & Euclid





"And another thing . . . I want you to be more assertive! I'm tired of everyone calling you Alexander the Pretty-Good!"

See Edstem post about 1-1 meetings with TAs not about current HW

Last class: Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

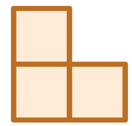
   $P(k)$ is true"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

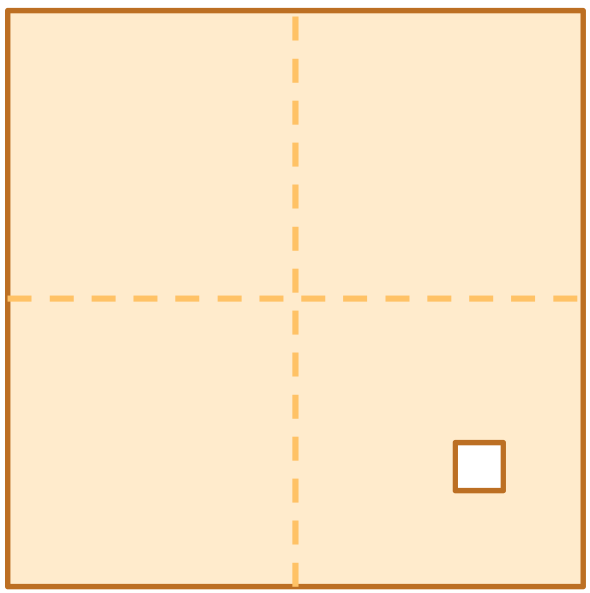
   Use the goal to figure out what you need.

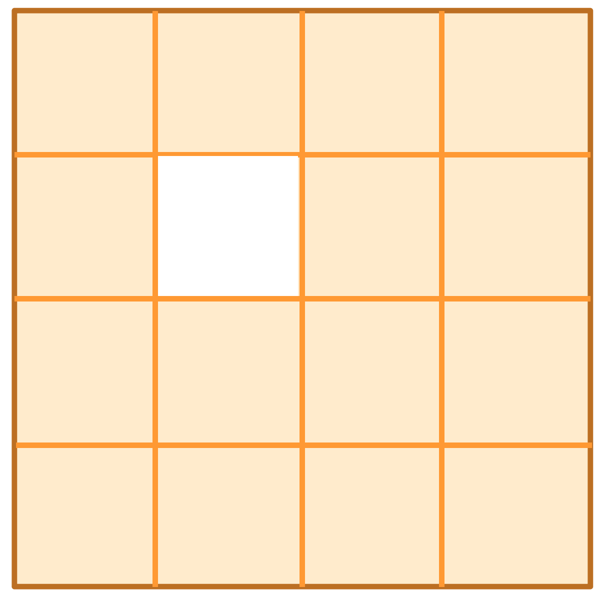
   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)

5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

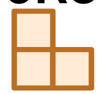
Checkerboard Tiling

- Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 





Checkerboard Tiling

1. **Let** P(n) **be "Any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **" .**
   We prove P(n) **for all** $n \geq 1$ **by induction on** n.

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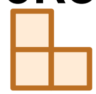
1. **Let** P(n) **be "Any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **" .**
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2. **Base Case:** n=1 

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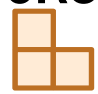
1. **Let** $P(n)$ **be "Any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **" .**
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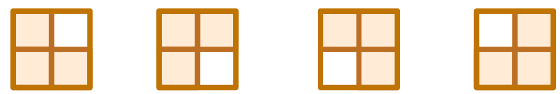
2. **Base Case:** $n=1$ 
3. **Inductive Hypothesis: Assume** $P(k)$ **for some arbitrary integer** $k \geq 1$

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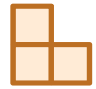
1. **Let** P(n) **be "Any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **".**
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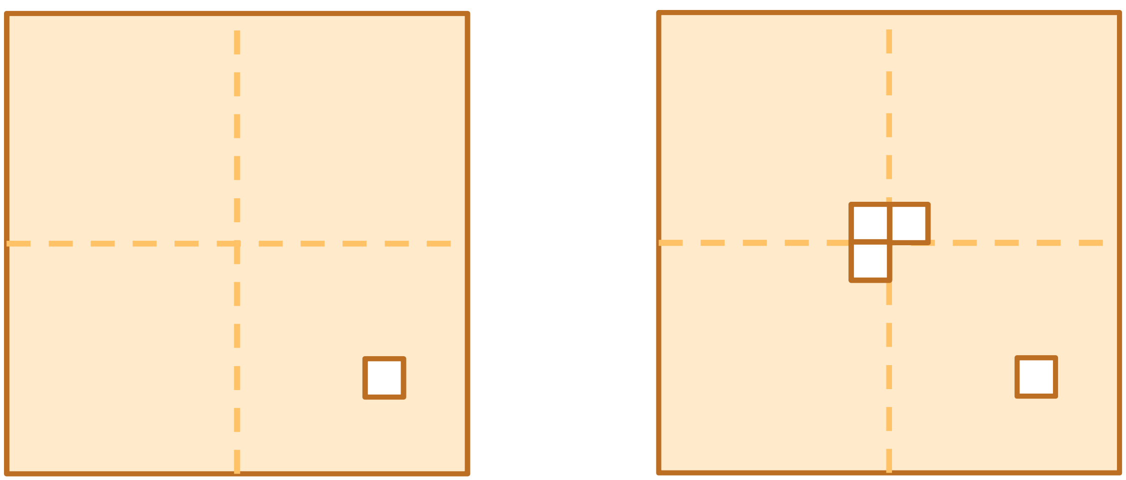
2. **Base Case:** n=1 
3. **Inductive Hypothesis: Assume** P(k) **for some arbitrary integer** $k \geq 1$
4. **Inductive Step: Prove** P(k+1)



Apply IH to each quadrant then fill with extra tile.

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

P(0)→P(1) P(1)→P(2) P(2)→P(3) P(3)→P(4) P(4)→P(5)

$$P(0) \quad P(1) \quad P(2) \quad P(3) \quad P(4) \quad P(5)$$

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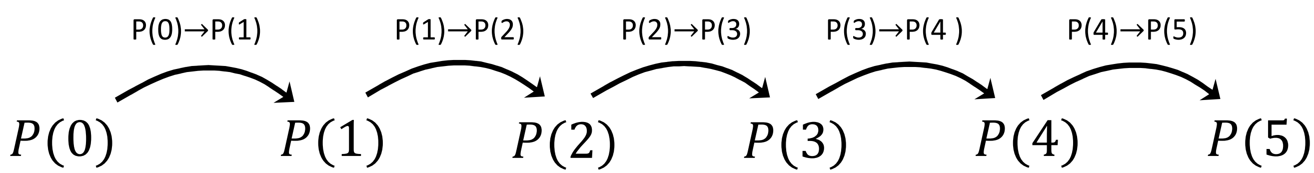

We made it harder than we needed to …

- **When we proved** $P(2)$ **we knew BOTH** $P(0)$ **and** $P(1)$
- **When we proved** $P(3)$ **we knew** $P(0)$ **and** $P(1)$ **and** $P(2)$
- **When we proved** $P(4)$ **we knew** $P(0)$, $P(1)$, $P(2)$, $P(3)$
- **etc.**

That's the essence of the idea of Strong Induction.

Strong Induction

$$\frac{P(0) \qquad \forall k\ \Big(\forall j\ \big(0 \le j \le k \rightarrow P(j)\big) \rightarrow P(k+1)\Big)}{\therefore \forall n\ P(n)}$$

Strong Induction

$$\frac{P(0) \qquad \forall k\, \Big(\forall j\, \big(0 \le j \le k \to P(j)\big) \to P(k+1)\Big)}{\therefore \forall n\, P(n)}$$

Strong induction for P **follows from ordinary induction for** Q **where**

$$Q(k) := \forall j\, \big(0 \le j \le k \to P(j)\big)$$

Note that $Q(0) = P(0)$ **and** $Q(k+1) \equiv Q(k) \wedge P(k+1)$ **and** $\forall n\, Q(n) \equiv \forall n\, P(n)$

Last class: Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(k)$ is true"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)

5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Strong Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ *is true for every integer* j *from* b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \dots, P(k)$ *are true) and point out where you are using it.*
   (Don't assume $P(k+1)$ *!!)*
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Recall: Fundamental Theorem of Arithmetic

Every integer > 1 has a unique prime factorization

48 = 2 • 2 • 2 • 2 • 3
591 = 3 • 197
45,523 = 45,523
321,950 = 2 • 5 • 5 • 47 • 137
1,234,567,890 = 2 • 3 • 3 • 5 • 3,607 • 3,803

We use strong induction to prove that a factorization into primes exists, but not that it is unique.

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   <u>Case:</u> <u>k+1</u> **<u>is composite:</u> Then** k+1=ab **for some integers** a **and** b
   where $2 \leq a, b \leq k$.

Every integer ≥ 2 is a product of (one or more) primes.

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4. **Inductive Step:**

Goal: Show $P(k+1)$; **i.e.** $k+1$ **is a product of primes**

<u>Case: k+1 is prime</u>: Then by definition $k+1$ **is a product of primes**

<u>Case: k+1 is composite:</u> Then $k+1=ab$ **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH,** $P(a)$ **and** $P(b)$ **are true so we have**

$a = p_1 p_2 \cdots p_r$ **and** $b = q_1 q_2 \cdots q_s$

for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.

Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

Since $k \geq 2$, **one of these cases must happen and so** $P(k+1)$ **is true.**

Every integer ≥ 2 is a product of (one or more) primes.

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   Since $k \geq 2$, **one of these cases must happen and so** P(k+1) **is true.**
5. **Thus** P(n) **is true for all integers** $n \geq 2$, **by strong induction.**

Strong Induction is particularly useful when…

…we need to analyze methods that on input k make a recursive call for an input different from $k-1$.

e.g.: Recursive Modular Exponentiation:

- For exponent $k > 0$ it made a recursive call with exponent $\mathrm{j} = k/2$ when k was even or $\mathrm{j} = k-1$ when k was odd.

Fast Exponentiation

```
public static int FastModExp(int a, int k, int modulus) {

        if (k == 0) {
            return 1;

         } else if ((k % 2) == 0) {
            long temp = FastModExp(a,k/2,modulus);
            return (temp * temp) % modulus;

        } else {
            long temp = FastModExp(a,k-1,modulus);
            return (a * temp) % modulus;
        }

}
```

$$a^{2j} \bmod m = \left(a^j \bmod m\right)^2 \bmod m$$

$$a^{2j+1} \bmod m = \left((a \bmod m) \cdot (a^{2j} \bmod m)\right) \bmod m$$

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We won't analyze this particular method by strong induction, but we could.

However, we will use strong induction to analyze other functions with recursive definitions.

Recursive definitions of functions

- $0! = 1$; $(n+1)! = (n+1) \cdot n!$ **for all** $n \geq 0$.
- $F(0) = 0$; $F(n+1) = F(n) + 1$ **for all** $n \geq 0$.
- $G(0) = 1$; $G(n+1) = 2 \cdot G(n)$ **for all** $n \geq 0$.
- $H(0) = 1$; $H(n+1) = 2^{H(n)}$ **for all** $n \geq 0$.

Prove $n! \leq n^n$ for all $n \geq 1$

1. **Let** P(n) **be** "n! ≤ n^n". **We will show that** P(n) **is true for all integers** n ≥ 1 **by induction.**
2. **Base Case** (n=1): $1!=1\cdot 0!=1\cdot 1=1=1^1$ **so** P(1) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 1. **I.e., suppose** $k! \leq k^k$.

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4. **Inductive Step:**

   Goal: Show P(k+1), **i.e. show** $(k+1)! \leq (k+1)^{k+1}$

   $(k+1)! = (k+1)\cdot k!$ **by definition of !**

   $\leq (k+1)\cdot k^k$ **by the IH**

   $\leq (k+1)\cdot (k+1)^k$ **since** $k \geq 0$

   $= (k+1)^{k+1}$

   Therefore P(k+1) **is true.**
5. **Thus** P(n) **is true for all** $n \geq 1$, **by induction.**

More Recursive Definitions

Suppose that $h: \mathbb{N} \to \mathbb{R}$.

Then we have familiar summation notation:

$\sum_{i=0}^{0} h(i) = h(0)$

$\sum_{i=0}^{n+1} h(i) = h(n+1) + \sum_{i=0}^{n} h(i)$ **for** $n \geq 0$

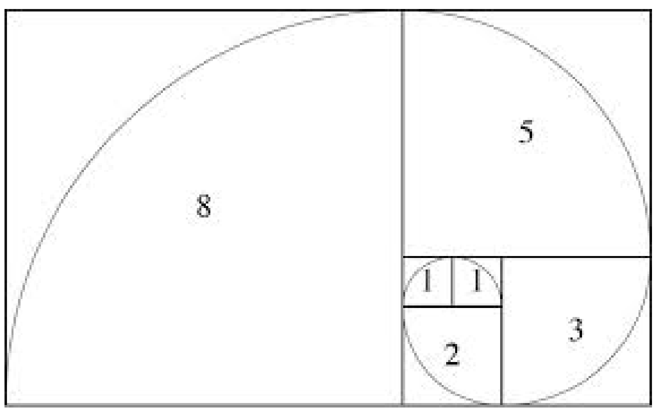
There is also product notation:

$\prod_{i=0}^{0} h(i) = h(0)$

$\prod_{i=0}^{n+1} h(i) = h(n+1) \cdot \prod_{i=0}^{n} h(i)$ **for** $n \geq 0$

Fibonacci Numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$ **for all** $n \geq 2$

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Tamás Görbe
@TamasGorbe

A Mathematician's Way* of Converting Miles to Kilometers

$$\begin{aligned} 3 \text{ mi} &\approx 5 \text{ km} \\ 5 \text{ mi} &\approx 8 \text{ km} \\ 8 \text{ mi} &\approx 13 \text{ km} \end{aligned}$$

f_n mi ≈ f_{n+1} km

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be** "$f_n < 2^n$". **We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**

$$f_0 = 0 \qquad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

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2. **Base Case:** $f_0=0 < 1= 2^0$ **so** P(0) **is true.**

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

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4. **Inductive Step:** **Goal: Show** P(k+1); **that is,** $f_{k+1} < 2^{k+1}$

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   Case k+1 = 1:

   Case k+1 ≥ 2:

$$f_0 = 0 \quad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

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   <u>Case k+1 = 1</u>: **Then** $f_1 = 1 < 2 = 2^1$ **so** P(k+1) **is true here.**

   <u>Case k+1 ≥ 2</u>:

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

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4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} < 2^{k+1}$

   Case k+1 = 1: Then $f_1 = 1 < 2 = 2^1$ so P(k+1) is true here.

   Case $k+1 \geq 2$: Then $f_{k+1} = f_k + f_{k-1}$ by definition

   $< 2^k + 2^{k-1}$ by the IH since $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k$

   $= 2^{k+1}$

   so P(k+1) is true in this case.

These are the only cases so P(k+1) follows.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0 = 0 < 1 = 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$**, we have** $f_j < 2^j$ **for every integer** j **from** 0 **to** k.
4. **Inductive Step:** **Goal: Show** P(k+1)**; that is,** $f_{k+1} < 2^{k+1}$

   Case k+1 = 1: **Then** $f_1 = 1 < 2 = 2^1$ **so** P(k+1) **is true here.**

   Case $k+1 \geq 2$: **Then** $f_{k+1} = f_k + f_{k-1}$ **by definition**

   $< 2^k + 2^{k-1}$ **by the IH since** $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) **is true in this case.**

   These are the only cases so P(k+1) **follows.**
5. **Therefore by strong induction,**

   $f_n < 2^n$ **for all integers** $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Inductive Proofs with Multiple Base Cases

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."
2. "Base Cases:" Prove $P(b)$, $P(b+1)$, ..., $P(c)$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary integer $k \geq c$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Inductive Proofs With Multiple Base Cases

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by ***strong*** induction."
2. "Base Cases:" Prove $P(b), P(b+1), \ldots, P(c)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq c$,

   $P(j)$ ***is true for every integer*** j ***from*** b **to** k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ ***are true) and point out where you are using it.***
   (Don't assume $P(k+1)$ ***!!)***
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Original Version

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0 = 0 < 1 = 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$**, we have** $f_j < 2^j$ **for every integer** j **from** 0 **to** k**.**
4. **Inductive Step:** **Goal: Show** P(k+1)**; that is,** $f_{k+1} < 2^{k+1}$

   Case k+1 = 1: **Then** $f_1 = 1 < 2 = 2^1$ **so** P(k+1) **is true here.**

   Case k+1 ≥ 2: **Then** $f_{k+1} = f_k + f_{k-1}$ **by definition**

   $< 2^k + 2^{k-1}$ **by the IH since** $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) **is true in this case.**

   These are the only cases so P(k+1) **follows.**

   First case in inductive step didn't need IH

5. **Therefore by strong induction,**

   $f_n < 2^n$ **for all integers** $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Multiple Base Case Version

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Cases: $f_0 = 0 < 1 = 2^0$ so P(0) is true.
   $f_1 = 1 < 2 = 2^1$ so P(1) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 1$, we have $f_j < 2^j$ for every integer j from 0 to k.
4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} < 2^{k+1}$

   We have $f_{k+1} = f_k + f_{k-1}$ by definition since $k+1 \geq 2$

   $< 2^k + 2^{k-1}$ by the IH since $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) is true.
5. Therefore, by strong induction, $f_n < 2^n$ for all integers $n \geq 0$.

Two base cases

Largest base case

Smallest base case

Two base cases, and two previous values used

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.

Two base cases, and two previous values used

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2 - 1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Cases:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2 - 1} = 2^0 = 1$ **so** P(2) **holds**
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ **so** P(3) **holds**

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2-1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Cases:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **holds**
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ **so** P(3) **holds**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 3$, P(j) **is true for every integer** j **from** 2 **to** k.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2-1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Cases:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **holds**
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ **so** P(3) **holds**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 3$, P(j) **is true for every integer** j **from** 2 **to** k**.**
4. **Inductive Step:** **Goal: Show** P(k+1)**; that is,** $f_{k+1} \geq 2^{(k+1)/2-1}$

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** $P(n)$ **be "**$f_n \geq 2^{n/2-1}$**". We prove that** $P(n)$ **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Cases:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** $P(2)$ **holds**
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ **so** $P(3)$ **holds**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 3$, $P(j)$ **is true for every integer** j **from** 2 **to** k**.**
4. **Inductive Step:** **Goal: Show** $P(k+1)$**; that is,** $f_{k+1} \geq 2^{(k+1)/2-1}$

   We have $f_{k+1} = f_k + f_{k-1}$ **by definition since** $k+1 \geq 2$

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ **by the IH since** $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   so $P(k+1)$ **is true.**
5. **Therefore by strong induction,** $f_n \geq 2^{n/2-1}$ **for all integers** $n \geq 2$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Why does this help us bound the running time of Euclid's Algorithm?

We already proved that $f_n \geq 2^{n/2 - 1}$ so $f_{n+1} \geq 2^{(n-1)/2}$

Therefore: if Euclid's Algorithm takes n steps
for $\gcd(a, b)$ with $a \geq b > 0$
then $a \geq 2^{(n-1)/2}$

so $(n-1)/2 \leq \log_2 a$ or $n \leq 1 + 2\log_2 a$
i.e., # of steps ≤ 1 + twice the # of bits in a.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes** n **steps for** $\gcd(a, b)$ **with** $a \geq b > 0$. **Then,** $a \geq f_{n+1}$.

An informal way to get the idea: Consider an n **step** gcd **calculation starting with** r_{n+1}=a **and** r_n=b:

$$
\begin{aligned}
r_{n+1} &= q_n r_n + r_{n-1} \\
r_n &= q_{n-1} r_{n-1} + r_{n-2} \\
&\ \ \dots \\
r_3 &= q_2 r_2 + r_1 \\
r_2 &= q_1 r_1 + 0
\end{aligned}
$$

f_0

For all $k \geq 2$, $r_{k-1} = r_{k+1} \bmod r_k$

"Euclid's algorithm is slowest on Fibonacci numbers and it takes only n steps for $\gcd(f_{n+1}, f_n)$"

Now $r_1 \geq 1$ **and each** q_k **must be** ≥ 1. **If we replace all the** q_K's **by** 1 **and replace** r_1 **by** 1 **, we can only reduce the** r_k's. **After that reduction,** $r_k = f_k$ **for every** k.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

We go by strong induction on n.

Let P(n) **be** "gcd(a,b) **with** a ≥ b>0 **takes** n **steps** → $a \geq f_{n+1}$" **for all** n ≥ 1.

<u>Base Case</u>: n=1 **Suppose Euclid's Algorithm with** a ≥ b > 0 **takes** 1 **step.**

By assumption, $a \geq b \geq 1 = f_2$ **so** P(1) **holds.**

n=2 **Suppose Euclid's Algorithm with** a ≥ b > 0 **takes** 2 **steps.**

Then a = q b + r

b = q' r + 0 **for** r ≥ 1.

Since a ≥ b > 0, **we must have** q ≥ 1 **and** b ≥1 **so**

$a = qb + r \geq b + r \geq 1+1 = 2 = f_3$ **and** P(2) **holds**

<u>Induction Hypothesis</u>: **Suppose that for some integer** k ≥ 2, P(j) **is true for all integers** j **s.t.** 1 ≤ j ≤ k

Running time of Euclid's algorithm

<u>Induction Hypothesis</u>: Suppose that for some integer $k \geq 2$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

<u>Inductive Step</u>: **Goal: if** gcd(a,b) **with** $a \geq b>0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2.}$

Since $k \geq 2$, **if** gcd(a,b) **with** $a \geq b>0$ **takes** $k+1 \geq 3$ **steps, the first** 3 **steps of Euclid's algorithm on** a **and** b **give us**

$a = q\,b + r$

$b = q'\,r + r'$

$r = q''\,r' + r''$

and there are k-2 **more steps after this. Note that this means that the** gcd(b, r) **takes** k **steps and** gcd(r, r') **takes** k-1 **steps.**

So since k, $k-1 \geq 1$, **by the IH we have** $b \geq f_{k+1}$ **and** $r \geq f_k$.

Running time of Euclid's algorithm

Induction Hypothesis: Suppose that for some integer $k \geq 2$, $P(j)$ is true for all integers j s.t. $1 \leq j \leq k$

Inductive Step: Goal: if $\gcd(a,b)$ with $a \geq b>0$ takes $k+1$ steps, then $a \geq f_{k+2}$.

Since $k \geq 2$, if $\gcd(a,b)$ with $a \geq b>0$ takes $k+1 \geq 3$ steps, the first 3 steps of Euclid's algorithm on a and b give us

$$a = q\, b + r$$
$$b = q'\, r + r'$$
$$r = q''\, r' + r''$$

and there are $k-2$ more steps after this. Note that this means that the $\gcd(b, r)$ takes k steps and $\gcd(r, r')$ takes $k-1$ steps.

So since $k, k-1 \geq 1$, by the IH we have $b \geq f_{k+1}$ and $r \geq f_k$.

Also, since $a \geq b$, we must have $q \geq 1$.

So $a = q\, b + r \geq b + r \geq f_{k+1} + f_k = f_{k+2}$ as required. ■