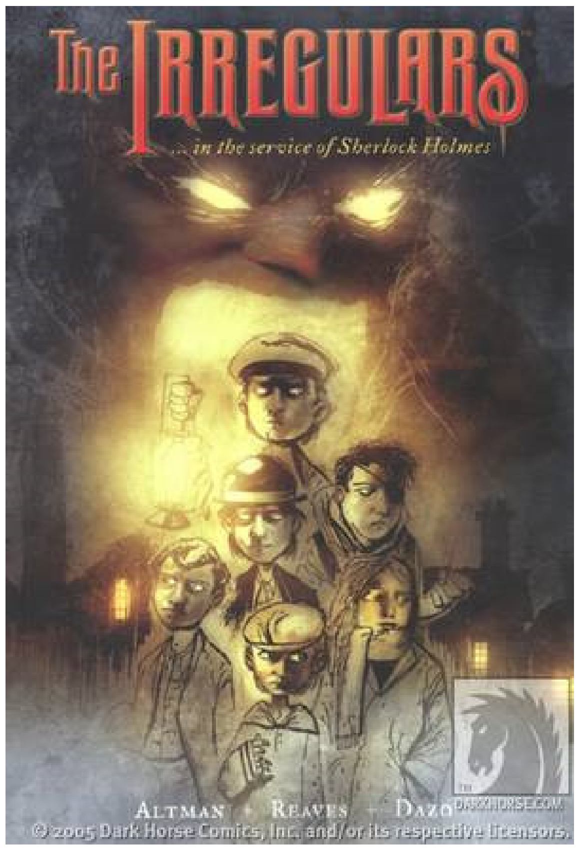
CSE 311: Foundations of Computing

Lecture 25: Languages vs Representations: Limitations of Finite Automata and Regular Expressions



Last time: Algorithms for Regular Languages

We have seen algorithms for

- RE to NFA
- NFA to DFA
- DFA/NFA to RE (not tested)
- DFA minimization

Practice three of these in HW.

(May also be on the final.)

Exponential Blow-up in Simulating Nondeterminism

- In general the DFA might need a state for every subset of states of the NFA
  - Power set of the set of states of the NFA
  - n-state NFA yields DFA with at most 2^n states
  - We saw an example where roughly 2^n is necessary

    "Is the n^{th} char from the end a 1?"

The famous "P=NP?" question asks whether a similar blow-up is always necessary to get rid of nondeterminism for polynomial-time algorithms

Applications of FSMs

- Implementation of regular expression matching in programs like `grep`
- Control structures for sequential logic in digital circuits
- Algorithms for communication and cache-coherence protocols
  - Each agent runs its own FSM
- Design specifications for reactive systems
  - Components are communicating FSMs

Applications of FSMs

- Formal verification of systems
  - Is an unsafe state reachable?
- Computer games Myst Adventure
  - FSMs provide worlds to explore
- Minimization algorithms for FSMs can be extended to more general models used in
  - Text prediction
  - Speech recognition

Application of FSMs: Pattern matching

- Given
  - a string s of n characters
  - a pattern p of m characters
  - usually $m \ll n$
- Find
  - all occurrences of the pattern p in the string s

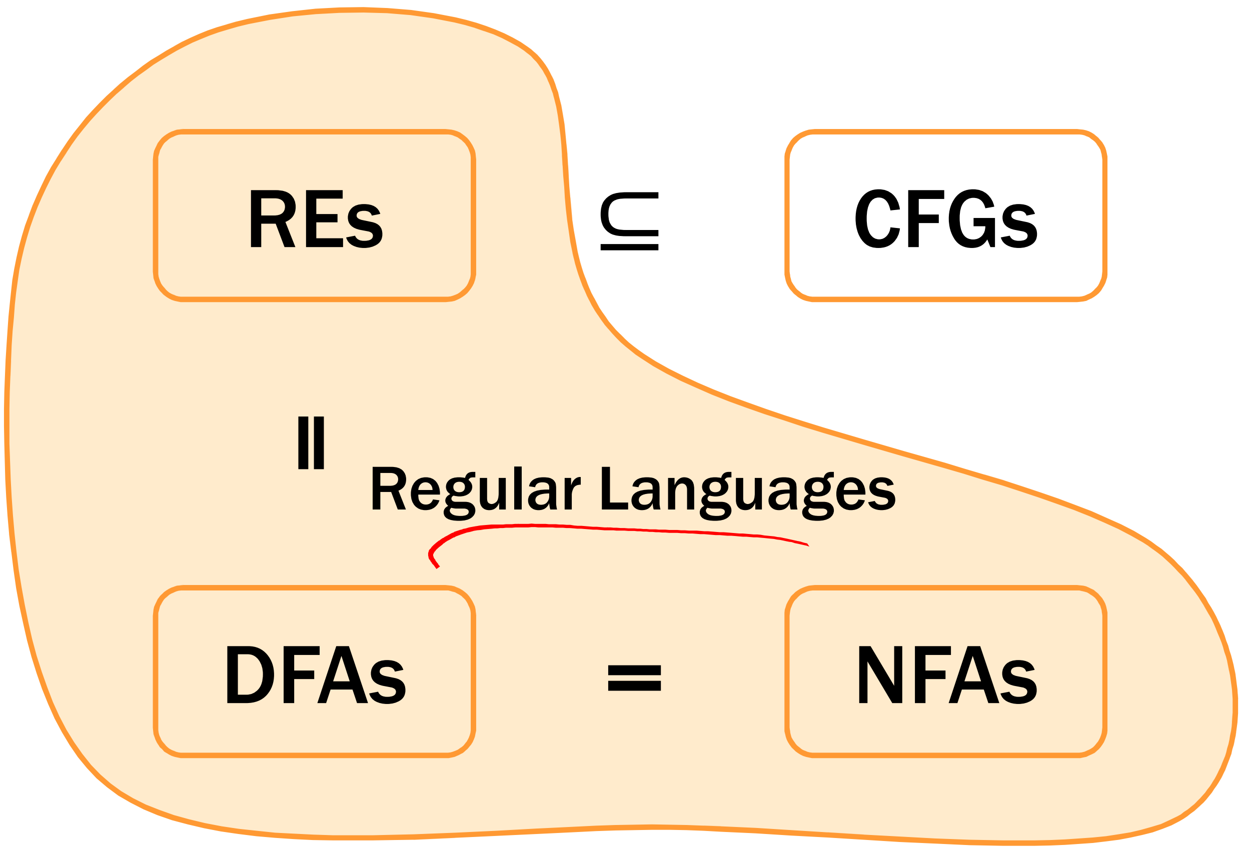
- Obvious algorithm:
  - try to see if p matches at each of the positions in s stop at a failed match and try matching at the next position: $O(mn)$ running time.

Application of FSMs: Pattern Matching

- With DFAs can do this in $O(m + n)$ time.

- See Extra Credit problem on HW8 for some ideas of how to get to $O(m^2 + n)$ time.

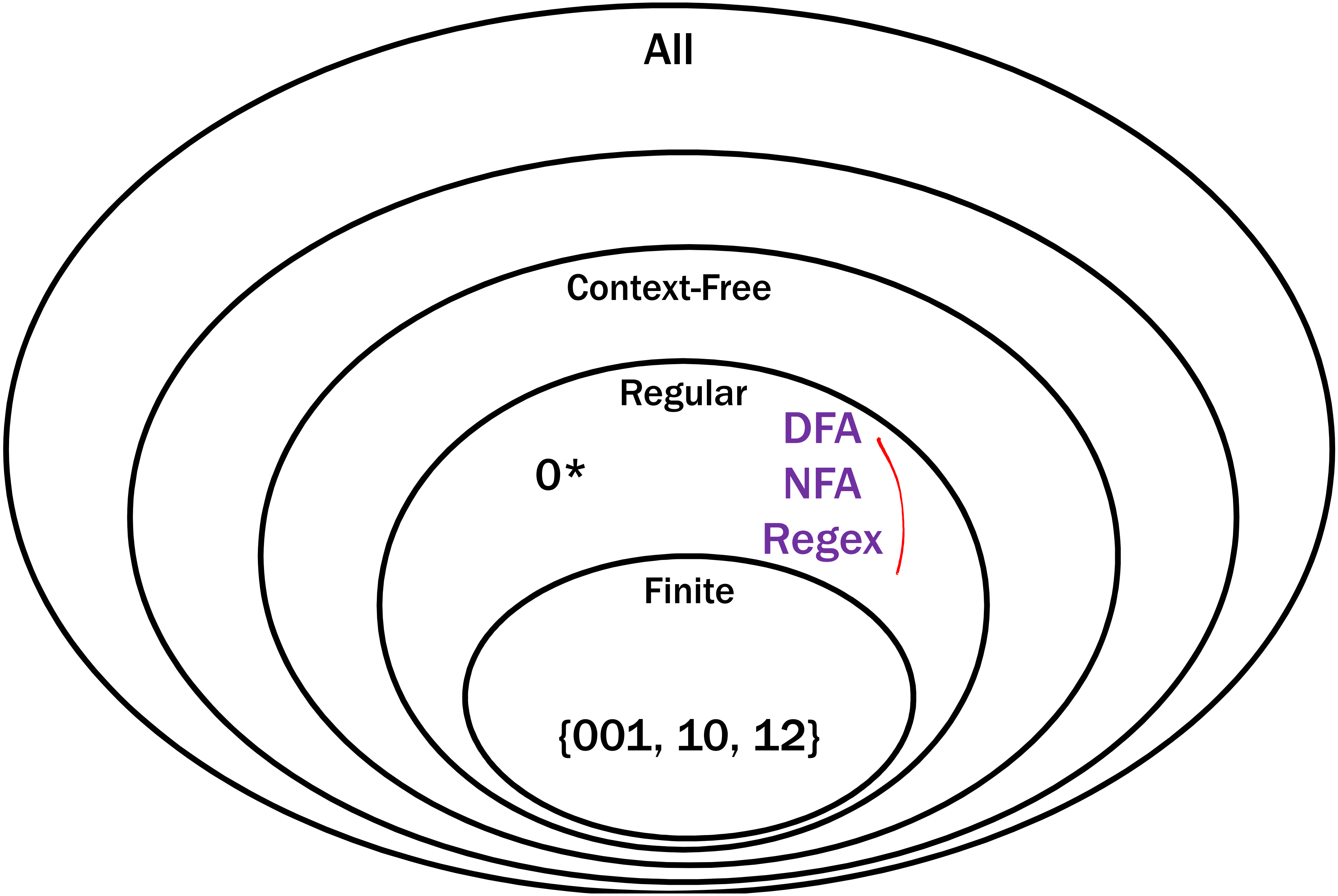
The story so far…



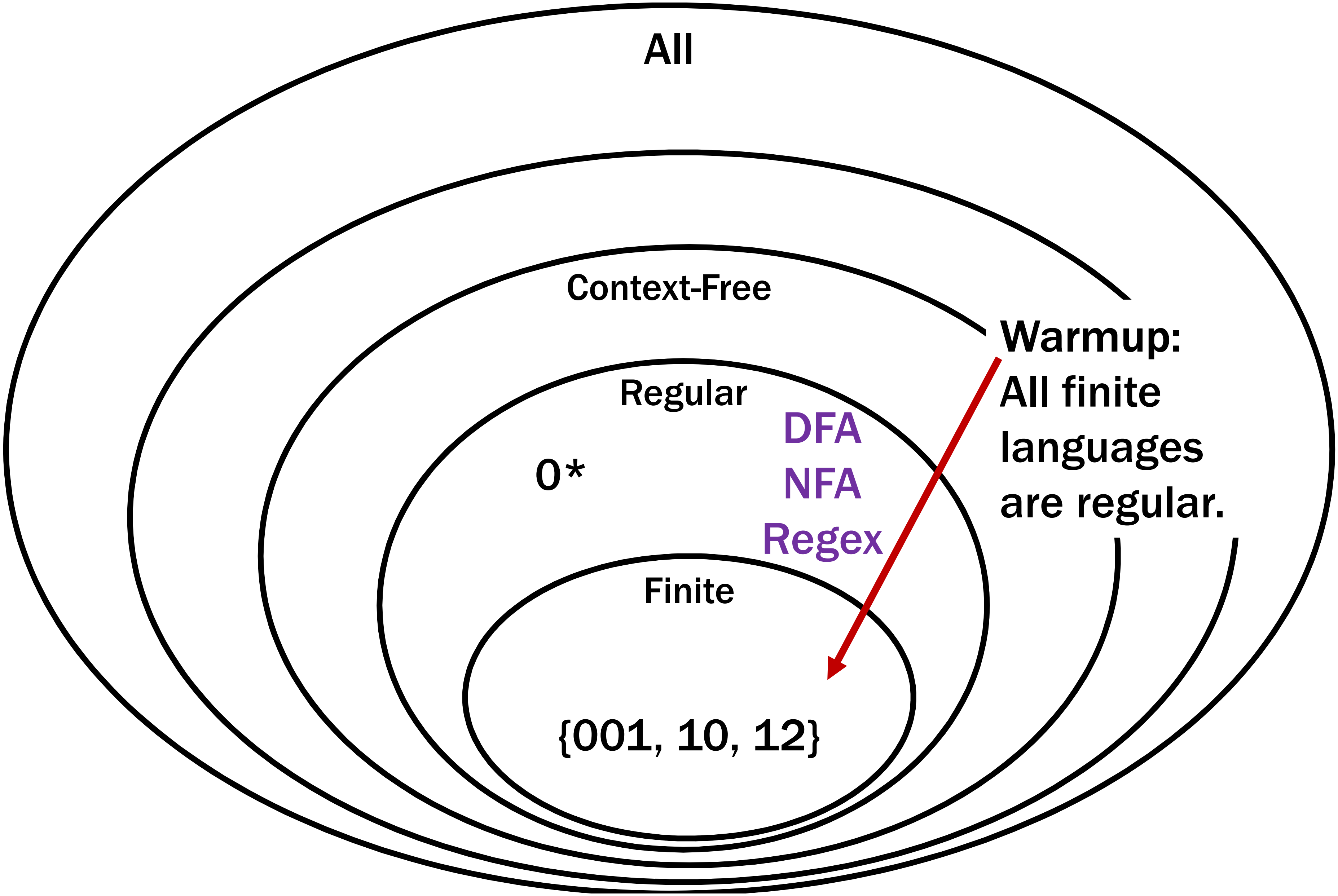
What languages have DFAs? CFGs?

All of them?

Languages and Representations!



Languages and Representations!



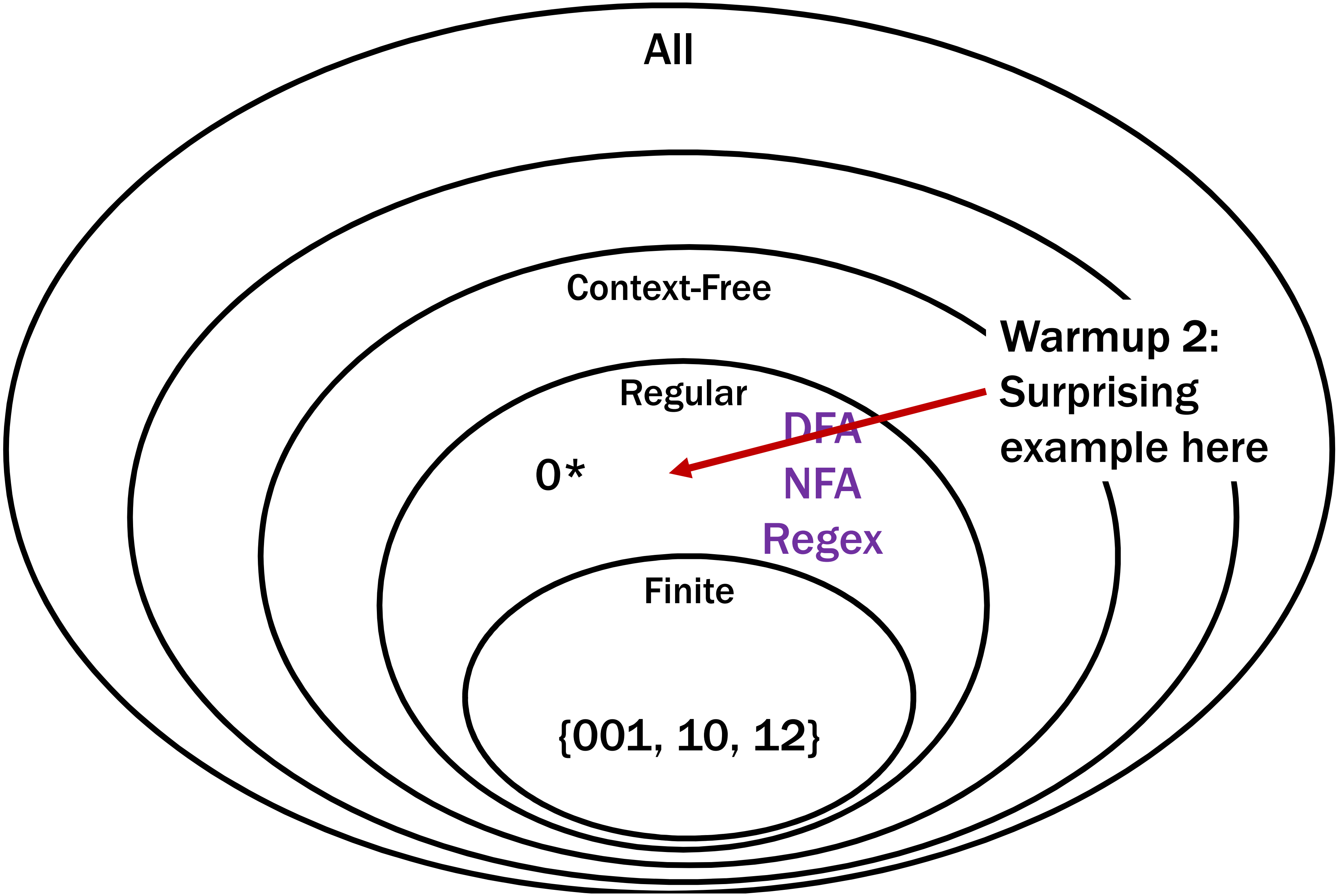
DFAs Recognize Any Finite Language

DFAs Recognize Any Finite Language

Construct a DFA for each string in the language.

Then, put them together using the union construction.

Languages and Machines!



An Interesting Infinite Regular Language

L = {x∈ {0, 1}*: x has an equal number of substrings 01 and 10}.

T T

L is infinite.

0, 00, 000, …

L is regular. How could this be?

01110010

That seems to require comparing counts…

- easy for a CFG
- but seems hard for DFAs!

~~0011~~

1001

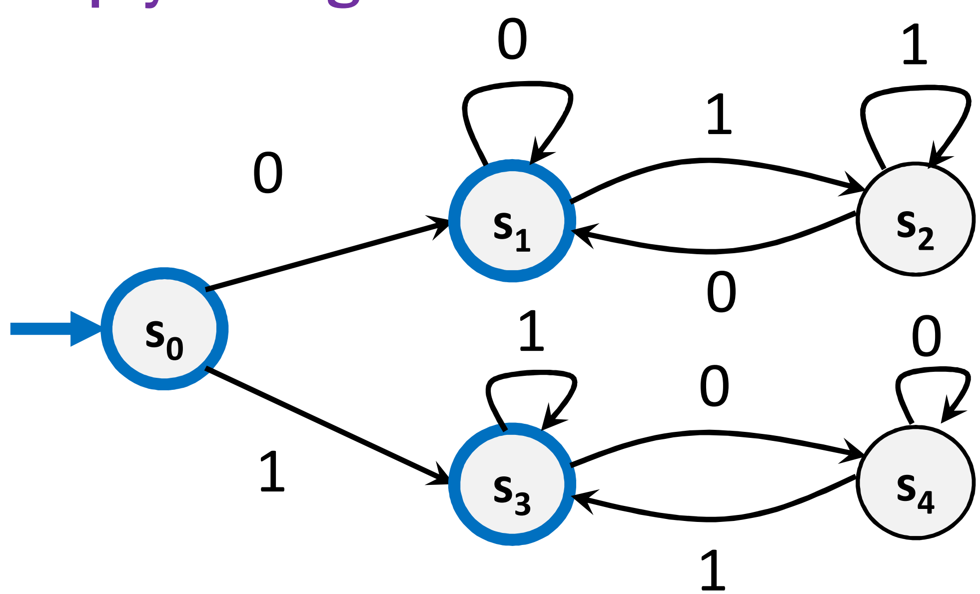
An Interesting Infinite Regular Language

L = {x∈ {0, 1}*: x has an equal number of substrings 01 and 10}.

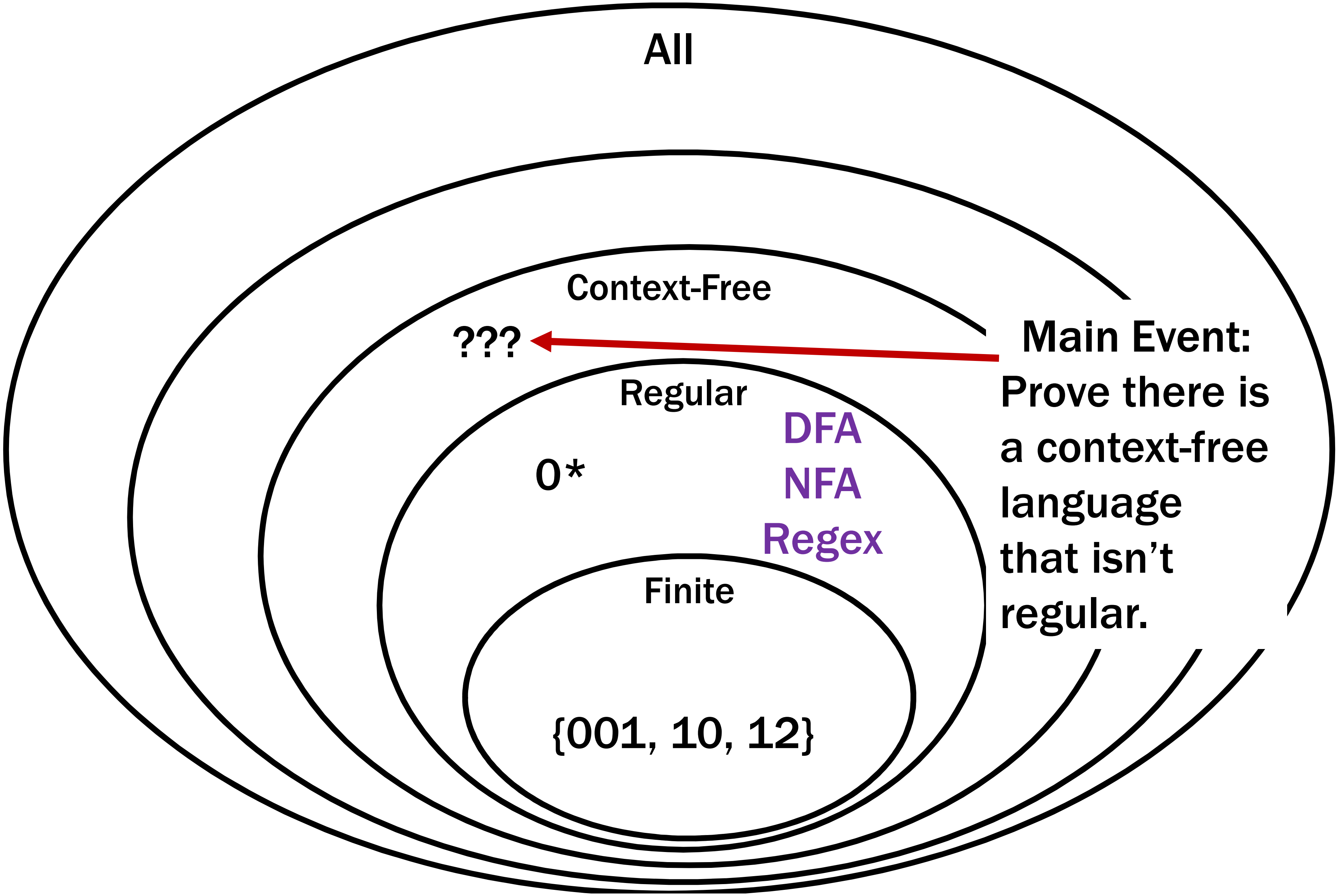
L is infinite.

0, 00, 000, ...

L is regular. How could this be? It is just the set of binary strings that are empty or begin and end with the same character!



Languages and Representations!



The language of "Binary Palindromes" is Context-Free

$$S \to \varepsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$$

1

0 1 0

0 0 1 0 0

0 0 0 1 0 0 0

Is the language of "Binary Palindromes" Regular ?

Intuition (NOT A PROOF!):

Q: What would a DFA need to keep track of to decide?

A: It would need to keep track of the "first part" of the input in order to check the second part against it

...but there are an infinite # of possible first parts and we only have finitely many states.

Proof idea: any machine that does not remember the entire first half will be wrong for some inputs

Part:

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- Assume (for contradiction) that some DFA (call it M) exists that recognizes B

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- Our goal is to show that M actually does not recognize B

How can a DFA fail to recognize B?

- when it accepts or rejects a string it shouldn't.

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- Assume (for contradiction) that some DFA (call it M) exists that recognizes B
- Our goal is to show that M actually does not recognize B, i.e., it accepts or rejects a string that it shouldn't

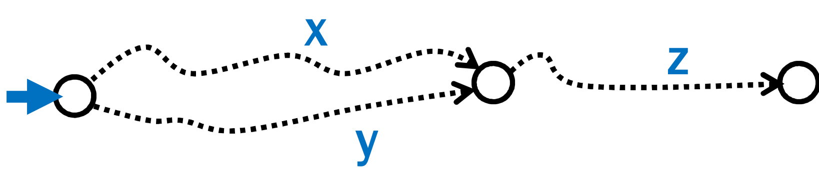
"M recognizes B" AND "M doesn't recognize B", which is a contradiction

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- Assume (for contradiction) that some DFA (call it M) exists that recognizes B
- We want to show: M accepts or rejects a string it shouldn't.

Key Idea 1: If two strings "collide" at any point, a DFA can no longer distinguish between them!



Both xz yz same answer for M

$x \bullet z = x_1\, x_2 \ldots x_n\, z_1\, z_2 \ldots z_k$

$y \bullet z = y_1\, y_2 \ldots y_m\, z_1\, z_2 \ldots z_k$

M is correct iff $\forall z \in \Sigma^* \,(x \bullet z \in B \leftrightarrow y \bullet z \in B)$

M is incorrect iff $\exists z \in \Sigma^* \,(x \bullet z \in B \not\leftrightarrow y \bullet z \in B)$

Should be different answers

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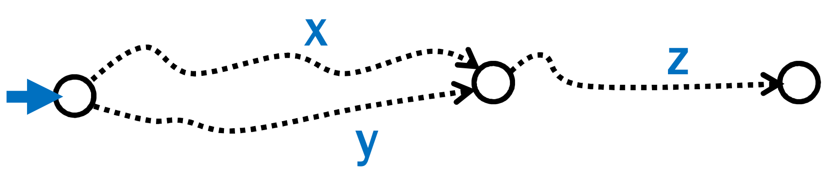


Key Idea 2: Our machine M has a finite number of states which means if we have *infinitely many* strings, two of them must collide!

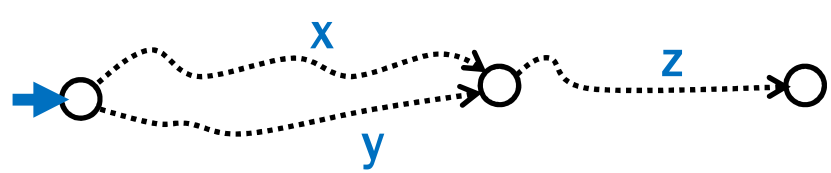
B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- Assume (for contradiction) that
  some DFA (call it M) exists that recognizes B
- We want to show: M accepts or rejects a string it shouldn't.

We choose an INFINITE set S of prefixes
(which we intend to complete later).

x,y	z
1	______
01	______
001	______
0001	______
00001	______
....	



$1, 01, 001, 0001, 00001, \ldots, 0^n1,$ ______

B = {binary palindromes} can't be recognized by any DFA

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- We want to show: M accepts or rejects a string it shouldn't.

We choose an INFINITE set S of prefixes (which we intend to complete later). It is critical that for *every pair* of strings in our set there is an <u>"accept" completion</u> that the two strings DO NOT SHARE.

x,y z

1_____

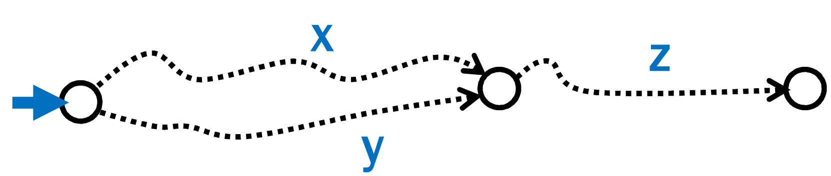
01 0000 ✗

001_____

0001_____

00001 0000 ✓

....



B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, recognizes B.

We show M accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, \ldots\} = \{0^n 1 : n \geq 0\}$.

Key Idea 2: Our machine has a finite number of states which means if we have infinitely many strings, two of them must collide!

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Consider $S = \{1, 01, 001, 0001, 00001, \ldots\} = \{0^n1 : n \geq 0\}$.

Since there are finitely many states in M and infinitely many strings in S, there exist strings $0^a1 \in S$ and $0^b1 \in S$ with $a \neq b$ that end in the same state of M.

SUPER IMPORTANT POINT: You do not get to choose what a and b are. Remember, we've just proven they exist…we must take the ones we're given!

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, accepts B.

We show M accepts or rejects a string it shouldn't.

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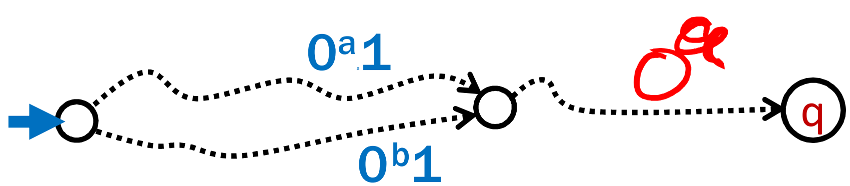
Now, consider appending 0^a *to both strings.*

Key Idea 1: If two strings "collide" at any point, a DFA can no longer distinguish between them!



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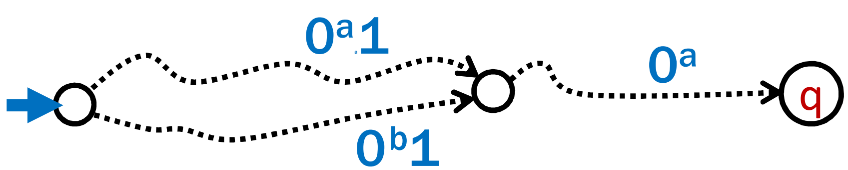

Then, since 0^a1 and 0^b1 end in the same state, 0^a10^a and 0^b10^a also end in the same state, call it q.

But then M makes a mistake: q needs to be an accept state since $0^a10^a \in B$, but M would accept $0^b10^a \notin B$ which is an error.

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, recognizes B.

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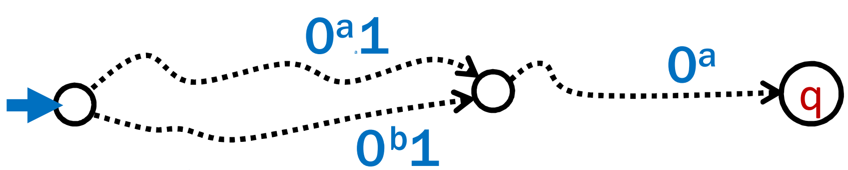

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This is a contradiction since we assumed that M recognizes B. Thus, no DFA recognizes B.

Showing that a Language L is not regular

1. "Suppose for contradiction that some DFA M recognizes L."
2. Consider an INFINITE set S of prefxes (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE.
3. "Since S is infinite and M has finitely many states, there must be two strings s_a and s_b in S for $s_a \neq s_b$ that end up at the same state of M."
4. Consider appending the (correct) completion t to each of the two strings.
5. "Since s_a and s_b both end up at the same state of M, and we appended the same string t, both $s_a t$ and $s_b t$ end at the same state q of M. Since $s_a t \in L$ and $s_b t \notin L$, M does not recognize L."
6. "Thus, no DFA recognizes L."

Prove $A = \{0^n1^n : n \geq 0\}$ is not regular

1. Suppose for contradiction that some DFA, **M**, recognizes **A**.

2. Let **S** = $\{0^n : n \geq 0\} = \{\varepsilon, 0, 00, 000, \ldots\}$

3. Since S is infinite, M has finitely many states
two strings in S collide for some $a \neq b$
$0^a, 0^b$

4. Let $t = 1^a$

5. Since $0^a, 0^b$ landed on same state of M
$0^a t = 0^a 1^a$ and $0^b t = 0^b 1^a$ land on same state: call it q.
$0^a 1^a \in A$ but $0^b 1^a \notin A$. M gives an incorrect answer on one of the two strings

Prove A = $\{0^n 1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M, recognizes A.

Let S = $\{0^n : n \geq 0\}$. Since S is infinite and M has finitely many states, there must be two strings, 0^a and 0^b for some $a \neq b$ that end in the same state in M.

Prove $A = \{0^n1^n : n \geq 0\}$ is not regular

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Consider appending 1^a to both strings.

Prove $A = \{0^n 1^n : n \geq 0\}$ is not regular

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Consider appending 1^a to both strings.

Note that $0^a 1^a \in A$, but $0^b 1^a \notin A$ since $a \neq b$. But they both end up in the same state of M, call it q. Since $0^a 1^a \in A$, state q must be an accept state but then M would incorrectly accept $0^b 1^a \notin A$ so M does not recognize A.

Thus, no DFA recognizes A.

Prove P = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, accepts P.

Let S = $\{ (^n : n \geq 0 \} = \{ \varepsilon, (, ((, (((, \ldots \}$

$(((()))(()))$

S infinite M has finite # of state

Some Two strings

$(^a , (^b$ for some $a \neq b$

land on same state of M

choose $t =)^a$

$(^a t = (^a)^a \in P$ and $(^b t = (^b)^a \notin P$

land in same state q

$(()((()))\,)^a$

Prove P = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, recognizes P.

Let S = { $(^n : n \geq 0$}. Since S is infinite and M has finitely many states, there must be two strings, $(^a$ and $(^b$ for some $a \neq b$ that end in the same state in M.

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Consider appending $)^a$ to both strings.

Note that $(^a)^a \in P$, but $(^b)^a \notin P$ since $a \neq b$. But they both end up in the same state of M, call it q. Since $(^a)^a \in P$, state q must be an accept state but then M would incorrectly accept $(^b)^a \notin P$ so M does not recognize P.

Thus, no DFA recognizes P.

Showing that a Language L is not regular

1. "Suppose for contradiction that some DFA M recognizes L."
2. Consider an INFINITE set S of prefixes (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE. (You need to come up with S.)
3. "Since S is infinite and M has finitely many states, there must be two strings s_a and s_b in S for $s_a \neq s_b$ that end up at the same state of M."
4. Consider appending the (hard) completion t to each of the two strings. (You need to come up with a hard t for s_a, s_b)
5. "Since s_a and s_b both end up at the same state of M, and we appended the same string t, both s_at and s_bt end at the same state q of M. Since $s_at \in L$ and $s_bt \notin L$, M does not recognize L."
6. "Thus, no DFA recognizes L."

Fact: This method is optimal

- Suppose that for a language L, the set S is a *largest* set of prefixes with the property that, for every pair $s_a \neq s_b \in S$, there is some string t such that one of $s_a t$, $s_b t$ is in L but the other isn't.
- If S is infinite, then L is not regular
- If S is finite, then the minimal DFA for L has precisely $|S|$ states, one reached by each member of S.

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- If S is infinite, then L is not regular
- If S is finite, then the minimal DFA for L has precisely $|S|$ states, one reached by each member of S.

Corollary: Our minimization algorithm was correct.

- we separated *exactly* those states for which some t would make one accept and another not accept

Important Notes

- It is not necessary for our strings xz with $x \in L$ to allow any string in the language
  - we only need to find a small "core" set of strings that must be distinguished by the machine

- It is not true that, if L is irregular and $L \subseteq U$, then U is irregular!
  - we always have $L \subseteq \Sigma^*$ and Σ^* is regular!
  - our argument needs different answers: $xz \in L \not\leftrightarrow yz \in L$

    for Σ^*, both strings are always in the language

Do not claim in your proof that, because $L \subseteq U$, U is also irregular

$\{01\} \subseteq \{0^n 1^n : n \geq 0\} \subseteq 0^* 1^*$

regular — not regular — regular