Quiz Section 6: Ordinary, Strong, and Structural Induction – Solutions

Task 1 – Midterm Review: Translation

Let your domain of discourse be all coffee drinks. You should use the following predicates:

- $\texttt{soy}(x)$ is true iff x contains soy milk.
- $\texttt{whole}(x)$ is true iff x contains whole milk.
- $\texttt{sugar}(x)$ is true iff x contains sugar
- $\texttt{decaf}(x)$ is true iff x is not caffeinated.
- $\texttt{vegan}(x)$ is true iff x is vegan.
- $\texttt{RobbieLikes}(x)$ is true iff Robbie likes the drink x.

Translate each of the following statements into predicate logic. You may use quantifiers, the predicates above, and usual math connectors like $=$ and $\neq$.

a) Coffee drinks with whole milk are not vegan.

$\forall x(\texttt{whole}(x) \rightarrow \neg\,\texttt{vegan}(x))$.

b) Robbie only likes one coffee drink, and that drink is not vegan.

$\exists x \forall y(\texttt{RobbieLikes}(x) \wedge \neg\,\texttt{Vegan}(x) \wedge [\texttt{RobbieLikes}(y) \rightarrow x = y])$
OR $\exists x(\texttt{RobbieLikes}(x) \wedge \neg\,\texttt{Vegan}(x) \wedge \forall y[\texttt{RobbieLikes}(y) \rightarrow x = y])$

c) There is a drink that has both sugar and soy milk.

$\exists x(\texttt{sugar}(x) \wedge \texttt{soy}(x))$

Translate the following symbolic logic statement into a (natural) English sentence. Take advantage of domain restriction.

$$\forall x([\texttt{decaf}(x) \wedge \texttt{RobbieLikes}(x)] \rightarrow \texttt{sugar}(x))$$

Every decaf drink that Robbie likes has sugar.

Statements like "For every decaf drink, if Robbie likes it then it has sugar" are equivalent, but only partially take advantage of domain restriction.

Task 2 – Casting Out Nines

Let $m \in \mathbb{N}$. This problem proves that if $9|m$, then the sum of the digits of m is a multiple of 9. (It actually proves a bit more.) In order to state this one needs the base 10 representation of m. Write $m = (d_n d_{n-1} \cdots d_1 d_0)_{10}$ where $d_0, \ldots, d_n$ are the base-10 digits of m; that is, each $d_0, \ldots, d_n \in \{0, 1, 2, \ldots, 9\}$ and $m = \sum_{i=0}^{n} d_i 10^i$.

Prove that casting out nines works for all $m \in \mathbb{N}$ by induction on the number of digits of m by showing that m and the sum of its digits are equivalent modulo 9. We can write this using summation notation as: Prove that for all $n \in \mathbb{N}$, $\sum_{i=0}^{n} d_i 10^i \equiv \sum_{i=0}^{n} d_i \pmod 9$ for all $d_0, \ldots, d_n \in \{0, 1, 2, \ldots, 9\}$. (In other words, prove that for all $n \in \mathbb{N}$, for all $d_0, \ldots, d_n \in \{0, 1, 2, \ldots, 9\}$,

$$d_0 + 10^1 \cdot d_1 + 10^2 \cdot d_2 + \cdots + 10^n \cdot d_n \equiv d_0 + d_1 + d_2 + \cdots + d_n \pmod 9).$$

Let $P(n)$ be "$\sum_{i=0}^{n} d_i 10^i \equiv \sum_{i=0}^{n} d_i \pmod 9$ for all $d_0, \ldots, d_n \in \{0, 1, 2, \ldots, 9\}$". We prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

Base Case ($n = 0$): Observe that $\sum_{i=0}^{0} d_i 10^i = d_0 10^0 = d_0 = \sum_{i=0}^{0} d_i$ since $10^0 = 1$. Therefore $\sum_{i=0}^{0} d_i 10^i \equiv \sum_{i=0}^{0} d_i \pmod 9$ and hence $P(0)$ holds.

Inductive Hypothesis: Assume that $P(k)$ is true for some arbitrary $k \in \mathbb{N}$. That is, we have $\sum_{i=0}^{k} d_i 10^i \equiv \sum_{i=0}^{k} d_i \pmod 9$ for all $d_0, \ldots, d_k \in \{0, 1, 2, \ldots, 9\}$.

Inductive Step: We now prove $P(k + 1)$:

Let $d_0, \ldots, d_{k+1} \in \{0, 1, 2, \ldots, 9\}$. We now have a couple of options for how to proceed:

Option 1: (Applying the IH using the larger terms.)

$$\sum_{i=0}^{k+1} d_i 10^i = d_0 + \sum_{i=1}^{k+1} d_i 10^i = d_0 + 10 \times \sum_{i=1}^{k+1} d_i 10^{i-1} = d_0 + 10 \times \sum_{j=0}^{k} d_{j+1} 10^j.$$

Consider the number with digits $d'_0 = d_1, \ldots, d'_k = d_{k+1}$. Then, by the inductive hypothesis we have $\sum_{j=0}^{k} d'_j 10^j \equiv \sum_{j=0}^{k} d'_j \pmod 9$, so we have $\sum_{j=0}^{k} d_{j+1} 10^j \equiv \sum_{j=0}^{k} d_{j+1} \pmod 9$ or equivalently, using the index $i = j + 1$,

$$\sum_{i=1}^{k+1} d_i 10^{i-1} \equiv \sum_{i=1}^{k+1} d_i \pmod 9.$$

Since $10 \equiv 1 \pmod 9$, by the product rule we have

$$10 \times \sum_{i=1}^{k+1} d_i 10^{i-1} \equiv \sum_{i=1}^{k+1} d_i \pmod 9$$

and so

$$\begin{aligned} \sum_{i=0}^{k+1} d_i 10^i \equiv d_0 + 10 \times \sum_{i=1}^{k+1} d_i 10^{i-1} &\equiv d_0 + \sum_{i=1}^{k+1} d_i \pmod 9 \\ &\equiv \sum_{i=0}^{k+1} d_j \pmod 9 \end{aligned}$$

and therefore $P(k+1)$ follows.

Option 2: (Apply the IH using the smaller terms, but need to apply it again.) Now by the inductive hypothesis we have

$$\sum_{i=0}^{k+1} d_i 10^i = \sum_{i=0}^{k} d_i 10^i + d_{k+1}10^{k+1} \equiv \sum_{i=0}^{k} d_i + d_{k+1}10^{k+1} \pmod 9.$$

It remains to handle the last term. We can figure that out again by the inductive hypothesis: Define $d'_k = d_{k+1}$ and $d'_j = 0$ for all $j \leqslant k$, Then by the inductive hypothesis we have

$$d_{k+1}10^k = d'_k 10^k = \sum_{i=0}^{k} d'_i 10^i \equiv \sum_{i=0}^{k} d'_i = d'_k = d_{k+1} \pmod 9.$$

Since $10 \equiv 1 \pmod 9$ we can use the multiplicative property of mod to obtain $d_{k+1}10^{k+1} \equiv d_{k+1} \pmod 9$. Plugging this into our previous line we have:

$$\begin{aligned}\sum_{i=0}^{k+1} d_i 10^i = \sum_{i=0}^{k} d_i 10^i + d_{k+1}10^{k+1} &\equiv \sum_{i=0}^{k} d_i + d_{k+1}10^{k+1} \pmod 9\\ &\equiv \sum_{i=0}^{k} d_i + d_{k+1} \pmod 9\\ &\equiv \sum_{i=0}^{k+1} d_i \pmod 9.\end{aligned}$$

Therefore $P(k+1)$ follows.

Conclusion: Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by induction.

Task 3 – In Harmony with Ordinary Induction

Define

$$H_i = \sum_{j=1}^{i} \frac{1}{j} = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

The numbers H_i are called the *harmonic* numbers.
Prove that $H_{2^n} \geqslant 1 + \frac{n}{2}$ for all integers $n \geqslant 0$.

Let $P(n)$ be "$H_{2^n} \geqslant 1 + \frac{n}{2}$". We will prove $P(n)$ for all integers $n \geqslant 0$ by induction.

Base Case $(n = 0)$**:** $H_{2^0} = H_1 = \sum_{j=1}^{1} \frac{1}{j} = 1 \geqslant 1 + \frac{0}{2}$, so $P(0)$ holds.

Induction Hypothesis: Assume that $H_{2^k} \geqslant 1 + \frac{k}{2}$ for some arbitrary integer $k \geqslant 0$.

Induction Step: Goal: Show $H_{2^{k+1}} \geqslant 1 + \frac{k+1}{2}$

$$
\begin{aligned}
H_{2^{k+1}} &= \sum_{j=1}^{2^{k+1}} \frac{1}{j} \\
&= \sum_{j=1}^{2^k} \frac{1}{j} + \sum_{j=2^k+1}^{2^{k+1}} \frac{1}{j} \\
&\geqslant 1 + \frac{k}{2} + \sum_{j=2^k+1}^{2^{k+1}} \frac{1}{j} && \text{[Induction Hypothesis]} \\
&\geqslant 1 + \frac{k}{2} + 2^k \cdot \frac{1}{2^{k+1}} && \text{[There are } 2^k \text{ terms in } [2^k+1, 2^{k+1}] \text{ and each is at least } \tfrac{1}{2^{k+1}}\text{]} \\
&\geqslant 1 + \frac{k}{2} + \frac{2^k}{2^{k+1}} \\
&\geqslant 1 + \frac{k}{2} + \frac{1}{2} \geqslant 1 + \frac{k+1}{2}
\end{aligned}
$$

So $P(k+1)$ follows.

Conclusion: $P(n)$ holds for all integers $n \geqslant 0$ by induction.

Task 4 – Induction with Formulas

These problems are a little more abstract.

a) i. Show that given two sets A and B that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. (Don't use induction.)

Let x be arbitrary. Then,

$$
\begin{aligned}
x \in \overline{A \cup B} &\equiv \neg(x \in A \cup B) && \text{[Definition of complement]} \\
&\equiv \neg(x \in A \vee x \in B) && \text{[Definition of union]} \\
&\equiv \neg(x \in A) \wedge \neg(x \in B) && \text{[De Morgan's Laws]} \\
&\equiv x \in \overline{A} \wedge x \in \overline{B} && \text{[Definition of complement]} \\
&\equiv x \in (\overline{A} \cap \overline{B}) && \text{[Definition of intersection]}
\end{aligned}
$$

Since x was arbitrary we have that $x \in \overline{A \cup B}$ if and only if $x \in \overline{A} \cap \overline{B}$ for all x. By the definition of set equality we've shown,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

ii. Show using induction that for an integer $n \geqslant 2$, given n sets $A_1, A_2, \ldots, A_{n-1}, A_n$ that

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}} \cap \overline{A_n}$$

Let $P(n)$ be "given n sets $A_1, A_2, \ldots, A_{n-1}, A_n$ it holds that $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}} \cap \overline{A_n}$." We show $P(n)$ for all integers $n \geqslant 2$ by induction on n.

Base Case: $P(2)$ says that for two sets A_1 and A_2 that $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$, which is exactly part (a) so $P(2)$ holds.

Inductive Hypothesis: Suppose that $P(k)$ holds for some arbitrary integer $k \geqslant 2$.

Inductive Step: Let $A_1, A_2, \ldots, A_k, A_{k+1}$ be sets. Then by part (a) we have,

$$\overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}} = \overline{A_1 \cup A_2 \cup \cdots \cup A_k} \cap \overline{A_{k+1}}.$$

By the inductive hypothesis we have $\overline{A_1 \cup A_2 \cup \cdots A_k} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}$. Thus,

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_k} \cap \overline{A_{k+1}} = (\overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_k}) \cap \overline{A_{k+1}}.$$

We've now shown

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}} = \overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_k} \cap \overline{A_{k+1}}.$$

which is exactly $P(k+1)$.

Conclusion $P(n)$ holds for all integers $n \geqslant 2$ by the principle of induction.

b) i. Show that given any integers a, b, and c, if $c \mid a$ and $c \mid b$, then $c \mid (a+b)$. (Don't use induction.)

Let a, b, and c be arbitrary integers and suppose that $c \mid a$ and $c \mid b$. Then by definition there exist integers j and k such that $a = jc$ and $b = kc$. Then $a + b = jc + kc = (j + k)c$. Since $j + k$ is an integer, by definition we have $c \mid (a + b)$.

ii. Show using induction that for any integer $n \geqslant 2$, given n numbers $a_1, a_2, \ldots, a_{n-1}, a_n$, for any integer c such that $c \mid a_i$ for $i = 1, 2, \ldots, n$, that

$$c \mid (a_1 + a_2 + \cdots + a_{n-1} + a_n).$$

In other words, if a number divides each term in a sum then that number divides the sum.

Let $P(n)$ be "given n numbers $a_1, a_2, \ldots, a_{n-1}, a_n$, for any integer c such that $c \mid a_i$ for $i = 1, 2, \ldots, n$, it holds that $c \mid (a_1 + a_2 + \cdots + a_n)$." We show $P(n)$ holds for all integer $n \geqslant 2$ by induction on n.

Base Case: $P(2)$ says that given two integers a_1 and a_2, for any integer c such that $c \mid a_1$ and $c \mid a_2$ it holds that $c \mid (a_1 + a_2)$. This is exactly part (a) so $P(2)$ holds.

Inductive Hypothesis: Suppose that $P(k)$ holds for some arbitrary integer $k \geqslant 2$.

Inductive Step: Let $a_1, a_2, \ldots, a_k, a_{k+1}$ be $k + 1$ integers. Let c be arbitrary and suppose that $c \mid a_i$ for $i = 1, 2, \ldots, k + 1$. Then we can write

$$a_1 + a_2 + \cdots + a_k + a_{k+1} = (a_1 + a_2 + \cdots + a_k) + a_{k+1}.$$

The sum $a_1 + a_2 + \cdots + a_k$ has k terms and c divides all of them, meaning we can apply the inductive hypothesis. It says that $c \mid (a_1 + a_2 + \cdots + a_k)$. Since $c \mid (a_1 + a_2 + \cdots + a_k)$ and $c \mid a_{k+1}$, by part (a) we have,

$$c \mid (a_1 + a_2 + \cdots + a_k + a_{k+1}).$$

This shows $P(k + 1)$.

Conclusion: $P(n)$ holds for all integers $n \geqslant 2$ by induction the principle of induction.

Task 5 – Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in year n is described by the function $f(n)$:

$$\begin{aligned} &f(0) = 0 \\ &f(1) = 1 \\ &f(n) = 2f(n-1) - f(n-2) \text{ for } n \geqslant 2 \end{aligned}$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

Let $P(n)$ be "$f(n) = n$". We prove that $P(n)$ is true for all $n \geqslant 0$ by strong induction on n.

Base Cases $(n = 0, n = 1)$**:** $f(0) = 0$ and $f(1) = 1$ by definition.

Inductive Hypothesis: Assume that $P(0), P(1), \ldots, P(k)$ all are true for some arbitrary $k \geqslant 1$.

Inductive Step: We show $P(k+1)$:

$$\begin{aligned} f(k+1) &= 2f(k) - f(k-1) && \text{[Definition of } f\text{]} \\ &= 2(k) - (k-1) && \text{[Induction Hypothesis]} \\ &= k+1 && \text{[Algebra]} \end{aligned}$$

Conclusion: $P(n)$ is true for all $n \in \mathbb{N}$ by principle of strong induction.

Task 6 – Strong Induction

Consider the function $a(n)$ defined for $n \geqslant 1$ recursively as follows.

$$a(1) = 1$$

$$a(2) = 3$$

$$a(n) = 2a(n-1) - a(n-2) \text{ for } n \geqslant 3$$

Use strong induction to prove that $a(n) = 2n - 1$ for all $n \geqslant 1$.

Let $P(n)$ be "$a(n) = 2n - 1$". We will show that $P(n)$ is true for all $n \geqslant 1$ by strong induction.

Base Cases $(n = 1, n = 2)$:
$(n = 1)$
$a(1) = 1 = 2 \cdot 1 - 1$

$(n = 2)$
$a(2) = 3 = 2 \cdot 2 - 1$

So, $P(1)$ and $P(2)$ hold.

Inductive Hypothesis:
Suppose that $P(k)$ is true for all integers $1 \leqslant j \leqslant k$ for some arbitrary $k \geqslant 2$.

Inductive Step:
We will show $P(k+1)$ holds.

$$\begin{aligned}
a(k+1) &= 2a(k) - a(k-1) && \text{[Definition of } a\text{]} \\
&= 2(2k-1) - (2(k-1) - 1) && \text{[Inductive Hypothesis]} \\
&= 2k + 1 && \text{[Algebra]} \\
&= 2(k+1) - 1 && \text{[Algebra]}
\end{aligned}$$

So, $P(k+1)$ holds.

Conclusion:
Therefore, $P(n)$ holds for all integers $n \geqslant 1$ by principle of strong induction.

Task 7 – Structural Induction

Define the set S as follows:
Basis Step: $[1,1,0] \in S$ and $[0,1,1] \in S$.
Recursive Step: If $[u,v,w] \in S$ and $[u',v',w'] \in S$
and $\alpha \in \mathbb{R}$ then $[\alpha u, \alpha v, \alpha w] \in S$ and $[u+u', v+v', w+w'] \in S$.

Prove that every $x \in S$ can be written in the form $x = [u,v,w]$ where $u,v,w \in \mathbb{R}$ and $v = u + w$.

Define $P(x)$ be "x is of the form $[u,v,w]$ where $u,v,w \in \mathbb{R}$ and $v = u + w$". We prove $P(x)$ for all $x \in S$ by structural induction.

Base Case: $0, 1 \in \mathbb{R}$ and $1 = 0 + 1$ and $1 = 1 + 0$, so $[0,1,1]$ and $[1,1,0]$ are of the form $[u,v,w]$ where $u,v,w \in \mathbb{R}$ and $v = u + w$ so $P([0,1,1])$ and $P([1,1,0])$ both holds

Inductive Hypothesis: Suppose that for some arbitrary $x \in S$ and $x' \in S$, $P(x)$ and $P(x')$ both are true.
Then $x = [u,v,w]$ and $x' = [u',v',w']$ for some $u,v,w,u',v',w' \in \mathbb{R}$ such that $v = u + w$ and $v' = u' + w'$.

Inductive Step: **Goal**: Prove that for all $\alpha \in \mathbb{R}$, we have $P([\alpha u, \alpha v, \alpha w])$ and $P([u+u', v+v', w+w'])$.

Clearly, by the closure of $\mathbb{R}$ under multiplication and addition, both $[\alpha u, \alpha v, \alpha w]$ and $[u+u', v+v', w+w']$ are of the right form. Finally, we observe that

$$\begin{aligned} \alpha v &= \alpha(u+w) && \text{by the I.H. applied to } x = [u,v,w] \\ &= \alpha u + \alpha w, \end{aligned}$$

so $P([\alpha u, \alpha v, \alpha w])$ follows and

$$\begin{aligned} v + v' &= u + w + v' && \text{by the I.H. applied to } x = [u,v,w] \\ &= u + w + u' + w' && \text{by the I.H. applied to } x' = [u',v',w'] \\ &= (u+w) + (w+w'), \end{aligned}$$

so $P([u+u', v+v', w+w'])$ follows.

Conclusion: Thus, $P(x)$ is true for all $x \in S$ by structural induction.