Quiz Section 8: CFGs, Relations, Graphs, and FSMs – Solutions

Task 1 – CFGs

Give CFGs for each of the following languages.

"Document" all the non-start variables in your grammar with an English description of the set of strings it generates. (You do not need to document the start variable because it is documented by the problem statement.)

a) All binary strings that end in 00.

$$\mathbf{S} \rightarrow 0\mathbf{S} \mid 1\mathbf{S} \mid 00$$

b) All binary strings that contain at least three 1's.

$$\mathbf{S} \rightarrow \mathbf{TTT}$$
$$\mathbf{T} \rightarrow 0\mathbf{T} \mid \mathbf{T}0 \mid 1\mathbf{T} \mid 1$$

$\mathbf{T}$ generates all binary strings with at least one 1.

c) All strings over $\{0,1,2\}$ with the same number of 1s and 0s and exactly one 2.

Hint: Try modifying the grammar from lecture for binary strings with the same number of 1s and 0s. (You may need to introduce new variables in the process.)

$$\mathbf{S} \rightarrow 2\mathbf{T} \mid \mathbf{T}2 \mid \mathbf{ST} \mid \mathbf{TS} \mid 0\mathbf{S}1 \mid 1\mathbf{S}0$$
$$\mathbf{T} \rightarrow \mathbf{TT} \mid 0\mathbf{T}1 \mid 1\mathbf{T}0 \mid \varepsilon$$

$\mathbf{T}$ is the grammar from lecture. It generates all binary strings with the same number of 1s and 0s.

Task 2 – Good, Good, Good, Good Relations

Each part below defines a relation R on a set. For each part, first state whether R is reflexive, symmetric, antisymmetric, and/or transitive. Second, if a relation does *not* have a property, then state a counterexample. (If a relation *does* have a property, you don't need to do anything other than saying so.)

a) Let $R = \{(x, y) \ : \ x = y + 1\}$ on $\mathbb{N}$.

> not reflexive (counterexample: $(1, 1) \notin R$), not symmetric (counterexample: $(2, 1) \in R$ but $(1, 2) \notin R$), antisymmetric, not transitive (counterexample: $(3, 2) \in R$ and $(2, 1) \in R$, but $(3, 1) \notin R$

b) Let $R = \{(x, y) \ : \ x^2 = y^2\}$ on $\mathbb{R}$.

> reflexive, symmetric, not antisymmetric (counterexample: $(-2, 2) \in R$ and $(2, -2) \in R$ but $2 \neq -2$), transitive

Task 3 – Relations

Let A be a set, and let R and S be relations on A. Suppose that R is reflexive.

a) Prove that $R \cup S$ is reflexive.

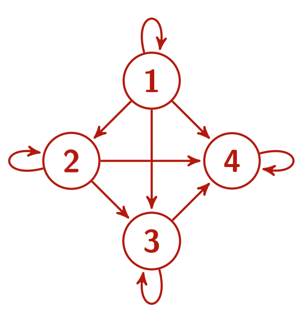
> Let x be arbitrary. Since we were given that R is reflexive, we know that $(x, x) \in R$. Thus it is also the case that $(x, x) \in R$ or $(x, x) \in S$ By definition of union, we know then that $(x, x) \in R \cup S$. Since x was arbitrary, by definition of reflexivity, we have shown that $R \cup S$ is reflexive.

b) Prove that $R \subseteq R^2$. (Remember that R^2 is defined to be $R \circ R$.)

> Let x and y be arbitrary. Suppose $(x, y) \in R$. Since R is reflexive, we know $(y, y) \in R$ as well. In other words, there is a z (namely y) such that $(x, z) \in R$ and $(z, y) \in R$. So by definition of relation composition, it follows that $(x, y) \in R \circ R = R^2$. Since x and y were arbitrary, by definition of subset $R \subseteq R^2$.

Task 4 – Closure

Draw the transitive-reflexive closure of $\{(1, 2), (2, 3), (3, 4)\}$ as a directed graph. We have drawn the vertices for you.



Task 5 – String Relations

Let $\Sigma = \{0,1\}$. Define the relation R on Σ^* by $(x,y) \in R$ if and only if $\text{len}(xy)$ is even. (Here xy is notation for the concatenation of the two strings x and y and len refers to the length of the string.)

Hint: In your proofs below, you may use the fact from lecture that $\text{len}(xy) = \text{len}(x) + \text{len}(y)$.

a) Prove that R is reflexive.

Let $a \in \Sigma^*$ be arbitrary. Let $n = \text{len}(a)$. Then $\text{len}(aa) = 2\text{len}(a) = 2n$, which is even by definition. Therefore $(a,a) \in R$ by definition of R. Since a was arbitrary, it follows by definition of reflexivity that R is reflexive.

b) Prove that R is symmetric.

Let a and b be arbitrary elements of Σ^*. Suppose $(a,b) \in R$. By definition of R, this means that $\text{len}(ab)$ is even. But we also have that

$$\begin{aligned}
\text{len}(ab) &= \text{len}(a) + \text{len}(b) && \text{from the fact from lecture} \\
&= \text{len}(b) + \text{len}(a) \\
&= \text{len}(ba) && \text{from the fact from lecture}
\end{aligned}$$

Therefore $\text{len}(ba)$ is also even. By definition of R it follows that $(b,a) \in R$. Since a and b were arbitrary, by definition of symmetry, it follows that R is symmetric.

c) Prove that R is transitive.

Let a, b, and c be arbitrary elements of Σ^*. Suppose $(a,b) \in R$ and $(b,c) \in R$. Then by definition of R, we have that $\text{len}(ab)$ is even and $\text{len}(bc)$ is even. Then

$$\begin{aligned}
\text{len}(ac) &= \text{len}(a) + \text{len}(c) && \text{from the fact from lecture} \\
&= (\text{len}(a) + \text{len}(b)) + (\text{len}(b) + \text{len}(c)) - 2\text{len}(b) && \text{algebra} \\
&= \text{len}(ab) + \text{len}(bc) - 2\text{len}(b) && \text{from the fact from lecture (twice)}
\end{aligned}$$

Thus $\text{len}(ac)$ is the sum of three terms: $\text{len}(ab)$, $\text{len}(bc)$, and $-2\text{len}(b)$. The first two terms are even by assumption, and the third is even by definition of even. Also, the sum of even numbers is even, so it follows that $\text{len}(ac)$ is even. By definition of R, it follows that $(a,c) \in R$. Since a, b, and c were arbitrary, it follows by definition of transitivity that R is transitive.

d) Is R antisymmetric? If so, prove it. If not, give a counterexample.

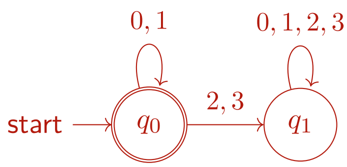
No. Counterexample: $(0,1) \in R$ since $\text{len}(01) = 2$ is even; also $(1,0) \in R$ since $\text{len}(10) = 2$ is even; but $0 \neq 1$.

Task 6 – DFAs, Stage 1

Let $\Sigma = \{0, 1, 2, 3\}$. Construct DFAs to recognize each of the following languages.

For all states in your DFA, include "documentation" for them by describing, in English, the set of strings that *end* in that state.

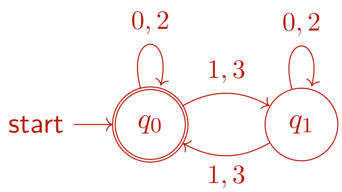
a) All binary strings.



q_0**:** all binary strings

q_1**:** strings that contain a character that is not 0 or 1.

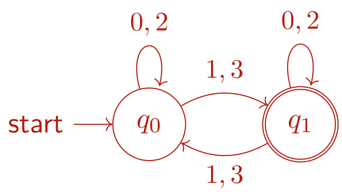
b) All strings whose digits sum to an even number.



q_0**:** all strings whose digits sum to an even number

q_1**:** all strings whose digits sum to an odd number

c) All strings whose digits sum to an odd number.



q_0**:** all strings whose digits sum to an even number

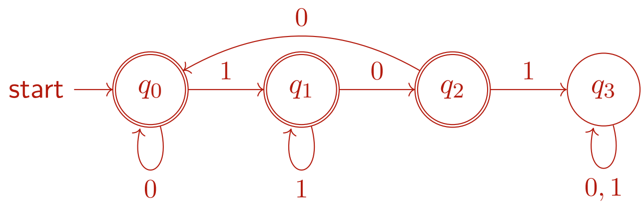
q_1**:** all strings whose digits sum to an odd number

Task 7 – DFAs, Stage 2

Let $\Sigma = \{0, 1\}$. Construct DFAs to recognize each of the following languages.

For all states in your DFA, include "documentation" for them by describing, in English, the set of strings that *end* in that state.

a) All strings that do not contain the substring 101.



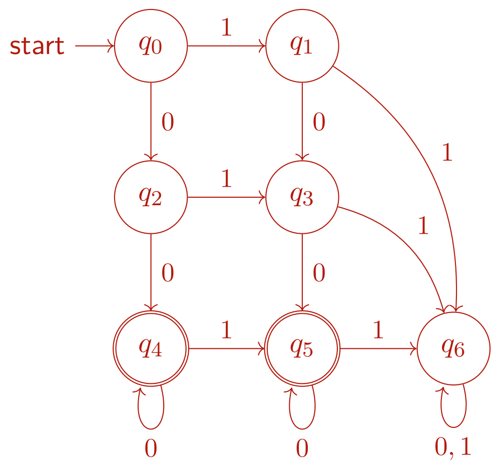
q_0: ε, 0, strings that don't contain 101 and end in 00. (Alternatively, strings that don't contain 101 whose end doesn't match any prefix of 101.)

q_1: strings that don't contain 101 and end in 1.

q_2: strings that don't contain 101 and end in 10.

q_3: string that contain 101.

b) All strings containing at least two 0's and at most one 1.



q_0: ε

q_1: 1

q_2: 0

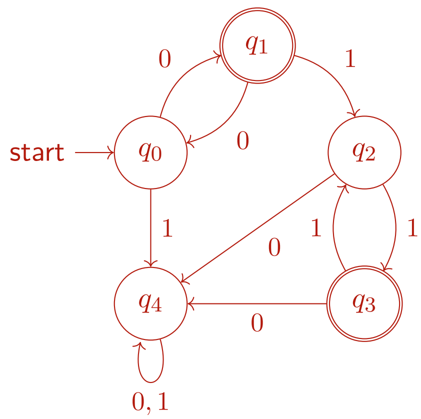
q_3: 01 or 10

q_4: strings consisting of only 0s of length at least 2

q_5: strings with exactly one 1 and at least two 0s

q_6: strings with at least two 1s

c) All strings containing an even number of 1's and an odd number of 0's and not containing the substring 10.



q_0**:** strings consisting of only 0s with even length (including ε)

q_1**:** strings consisting of only 0s with odd length

q_2**:** strings consisting of an odd number of 0s followed by an odd number of 1s

q_3**:** strings consisting of an odd number of 0s followed by an even number of 1s

q_4**:** strings that contain the substring 10 or that start with an even number of 0s followed by a 1