More English Proofs

CSE 311 Winter 24
Lecture 8

Regrades

TAs make mistakes!

When I was a TA, I made errors on 1 or 2% of my grading that needed to be corrected. If we made a mistake, file a regrade request on gradescope.

But those are only for mistakes, not for whether "-1 would be more fair"

If you are confused, please talk to us!

My favorite office hours questions are "can we talk about the best way to do something on the homework we just got back?"

If **after** you do a regrade request on gradescope, you still think a grading was incorrect, send email to Robbie.

Regrade requests will close about 1 week after homework is returned.

Integer

We need a basic starting point to be able to prove things.

Objects to work with.

An integer: is any real number with no fractional part.

Some definitions to analyze

Even

`Even(x)` := An integer, x, is even if and only if there is an integer k such that $x = 2k$.

Odd

`Odd(x)` := An integer, x, is odd if and only if there is an integer k such that $x = 2k + 1$.

Our First Direct Proof

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\,(\text{Even}(x) \to \text{Even}(x^2))$

Proof: Let x be an arbitrary integer. Suppose that x is even.

By definition of even, $x = 2k$ for some integer k.

Squaring both sides, we see that:

$$x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$$

Because k is an integer, $2k^2$ is also an integer.

So x^2 is two times an integer.

Which is exactly the definition of even, so x^2 is even.

Since x was an arbitrary integer, we conclude that for all integers x, if x is even then x^2 is also even.

Direct Proof Template

Prove: $\forall x \left(\mathrm{Even}(x) \rightarrow \mathrm{Even}(x^2)\right)$

Declare an arbitrary variable for each ∀.	Let x be an arbitrary integer.
Assume the left side of the implication.	Suppose that x is even.
Unroll the predicate definitions.	Then by definition of even, there exists some integer k such that $x = 2k$.
Manipulate towards the goal.	Squaring both sides, we see that: $x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$
Reroll definitions into the right side of the implication.	Because k is an integer, then $2k^2$ is also an integer. So x^2 is two times an integer. So by definition of even, x^2 is even.
Conclude that you have proved the claim.	Since x was an arbitrary integer, we can conclude that for all integers x, if x is even then x^2 is even.

Direct Proof Steps

These are the usual steps. We'll see different outlines in the future!!

- Introduction
  - Declare an arbitrary variable for each ∀ quantifier
  - Assume the left side of the implication
- Core of the proof
  - Unroll the predicate definitions
  - Manipulate towards the goal (using creativity, algebra, etc.)
  - Reroll definitions into the right side of the implication
- Conclude that you have proved the claim

Another Direct Proof

Prove: "The product of two odd integers is odd."

What's the claim in logic?

How would we prove this claim?

Another Direct Proof

Definitions
$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: "The product of two odd integers is odd."

$$\forall x \forall y \big((\text{Odd}(x) \land \text{Odd}(y)) \to \text{Odd}(xy)\big)$$

$$5 = 2 \cdot 2 + 1$$

Another Direct Proof

Prove: "The product of two odd integers is odd."

What's the claim in logic? $\forall x \forall y \left(\left(\mathrm{Odd}(x) \land \mathrm{Odd}(y) \right) \rightarrow \mathrm{Odd}(xy) \right)$

How would we prove this claim?

Direct Proof. In particular, we'll let x, y be arbitrary integers. We'll suppose x, y are odd. We'll show that $x \cdot y$ is odd.

Another Direct Proof

k_1, k_2

Definitions
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: "The product of two odd integers is odd."

$$\forall x \forall y \Big(\big(\mathrm{Odd}(x) \wedge \mathrm{Odd}(y)\big) \rightarrow \mathrm{Odd}(xy) \Big)$$

Let x and y be arbitrary integers. Suppose that x and y are odd. Then by definition of odd, there exists some integer k such that $x = 2k + 1$, and some integer j such that $y = 2j + 1$.

Then multiplying x and y, we can see that:

$$xy = (2k + 1) \cdot (2j + 1) = 4kj + 2j + 2k + 1 = 2(2kj + j + k) + 1$$

Since k, j are integers, $2kj + j + k$ is an integer. So by definition of odd, xy is odd.
Since x, y were arbitrary, we have shown that the product of two odd integers is odd.

A note on Domain of Discourse

"The product of two odd integers is odd."

Domain: Integers

Translation:

$\forall x \forall y \left(\left(\mathrm{Odd}(x) \land \mathrm{Odd}(y) \right) \rightarrow \mathrm{Odd}(xy) \right)$

Proof Outline:

Let x and y be arbitrary integers.

Suppose x and y are odd.

Show xy is odd.

Domain: Odd Integers

Translation:

$\forall x \forall y \left(\mathrm{Odd}(xy) \right)$

Proof Outline:

Let x and y be arbitrary odd integers.

Show xy is odd.

A note on Translation to Logic

- We first translate the claim to predicate logic because:
  - The translation makes it precise what we are proving
  - The translation hints at the structure of the proof

    e.g. for each ∀, introduce an arbitrary variable
- In practice, computer scientists identify the proof claim and structure without predicate logic translation
- Eventually we'll stop asking you to translate to logic first

Square

$9 = 3^2$

Definition:

An integer x is square iff there exists an integer k such that $x = k^2$.

$$\text{Square}(x) \coloneqq \exists k\ (x = k^2)$$

Yet Another Direct Proof

Definitions

$\text{Square}(x) \coloneqq \exists k\,(x = k^2)$

Prove: The product of two square integers is square.

What's the claim in logic?

$$\forall n \forall m \left(\left(\text{Square}(n) \land \text{Square}(m)\right) \to \text{Square}(nm) \right)$$

Prove this claim.

Yet Another Direct Proof

Definitions

$\text{Square}(x) \coloneqq \exists k\,(x = k^2)$

Prove: "The product of two square integers is square."

$$\forall n \forall m \Big(\big(\text{Square}(n) \land \text{Square}(m)\big) \rightarrow \text{Square}(nm) \Big)$$

Yet Another Direct Proof

Definitions

$\text{Square}(x) \coloneqq \exists k\, (x = k^2)$

Prove: "The product of two square integers is square."

$$\forall n \forall m \Big(\big(\text{Square}(n) \land \text{Square}(m)\big) \to \text{Square}(nm) \Big)$$

Let n and m be arbitrary integers. Suppose that n and m are square. Then by definition of square, $n = k^2$ for some integer k, and $m = j^2$ for some integer j.

Then multiplying n and m, we can see:

$nm = k^2 \cdot j^2 = (kj)^2$

Since k and j are integers, kj is an integer. So by definition of square, nm is square. Since n and m were arbitrary, we have shown that the product of two square integers is square.

More Proof Techniques

Proving an exists statement

How do I convince you $\exists x(P(x))$?

Show me the x! And convince me that $P(x)$ is true for that x.

Domain: Integers

Claim $\exists x \text{ Even}(x)$

Proof: Consider $x = 2$. We see that $2 = 2 \cdot 1$. Since 1 is an integer $2 = 2k$ for an integer k, which means 2 is even by definition, as required.

Two claims, two proof techniques

$$p \to q$$

Suppose I claim that all square numbers are even.

$$\neg[\neg p \vee q]$$

That…doesn't look right.

How do you prove me wrong?

$$p \wedge \neg q$$

What am I trying to prove? First write symbols for "¬(for all square numbers)"

$$\neg(\forall x\,[\text{Square}(x) \to \text{Even}(x)])$$

Then 'distribute' the negation sign.

$$\exists x\,(\neg[\text{Square}(x) \to \text{Even}(x)])$$

$$\exists x\,(\text{Square}(x) \wedge \neg\text{Even}(x))$$

Two claims, two proof techniques

Suppose I claim that all square numbers are even.

That…doesn't look right.

How do you prove me wrong?

Want to show: $\exists x(\texttt{Square}(x) \wedge \neg\texttt{Even}(x))$

Consider $x = 9$. 9 is a perfect square ($9 = 3^2$) and 9 is not even (since it is $2 \cdot 4 + 1$, i.e., odd).

Proof By [Counter]Example

To prove an existential statement (or disprove a universal statement), provide an example, and demonstrate that it is the needed example.

You don’t have to explain where it came from! (In fact, you **shouldn’t)**

Computer scientists and mathematicians like to keep an air of mystery around our proofs.

(or more charitably, we want to focus on just enough to believe the claim)

Skeleton of an Exists Proof

To show $\exists x(P(x))$

Consider $x =$[the value that will work]

[Show that x does cause $P(x)$ to be true.]

So [value] is the desired x.

You'll probably need some "scratch work" to determine what to set x to. That might not end up in the final proof!

Proof By Cases

Claim: $\forall x($ `Prime`$(x) \rightarrow [$ `Odd`$(x) \vee$ `PowerOfTwo`$(x)])$

Where `PowerOfTwo`$(x) :=\exists c($`Integer`$(c) \wedge x = 2\hat{}c)$

You may assume for this proof that 2 is the only even prime.

Let x be an arbitrary prime number.

You need two different arguments!

x is odd or x is a power of 2.

Proof By Cases

Claim: $\forall x($ `Prime`$(x) \rightarrow [$ `Odd`$(x) \vee$ `PowerOfTwo`$(x)])$

Where `PowerOfTwo`$(x) :=\exists c($`Integer`$(c) \wedge x = 2\text{^}c)$

You may assume for this proof that 2 is the only even prime.

Let x be an arbitrary prime number. We have two cases:

Case 1: x is even

Case 2:x is odd.

Proof By Cases

Claim: $\forall x($ `Prime` $(x) \rightarrow [$ `Odd` $(x) \vee$ `PowerOfTwo` $(x)])$

Where `PowerOfTwo` $(x) \coloneqq \exists c($`Integer`$(c) \wedge x = 2\hat{}c)$

You may assume for this proof that 2 is the only even prime.

Let x be an arbitrary prime number. We have two cases:

Case 1: x is even

2 is the only even prime. Since $2 = 2^1$, `PowerOfTwo` (2) is true, so `Odd` $(2) \vee$ `PowerOfTwo` (2) is also true.

Case 2: x is odd.

Then $Odd(x)$ is true, so Odd(x)∨PowerOfTwo(x) is also true.

In both cases, we conclude `Odd` $(x) \vee$ `PowerOfTwo` (x). Since x was arbitrary, we have that all prime numbers are odd or powers of two, as required.

Proof By Cases

Make it clear how you decide which case your in.

It should be obvious your cases are “exhaustive”

Reach the same conclusion in each of the cases, and you can say you’ve got that conclusion no matter what (outside the cases).

Advanced version: sometimes you end up arguing a certain case “can’t happen”