CSE 311 Section 3

Quantifiers and Proofs

Administrivia & Introductions

Announcements & Reminders

- HW1
  - If you think something was graded incorrectly, submit a regrade request!

- HW2 was due yesterday 1/17 on Gradescope
  - Use late days if you need them!
  - Gradescope: Make sure you select the pages for each question correctly

- HW3
  - Due Wednesday 1/24 @ 11:59pm

References

- Helpful reference sheets can be found on the course website!
  - https://courses.cs.washington.edu/courses/cse311/23au/resources/
- How to LaTeX (found on Assignments page of website):
  - https://courses.cs.washington.edu/courses/cse311/23au/assignments/HowToLaTeX.pdf
- Equivalence Reference Sheet
  - https://courses.cs.washington.edu/courses/cse311/23au/resources/reference-logical_equiv.pdf
  - https://courses.cs.washington.edu/courses/cse311/23au/resources/logicalConnectPoster.pdf
- Boolean Algebra Reference Sheet
  - https://courses.cs.washington.edu/courses/cse311/23au/resources/reference-boolean-alg.pdf
- Plus more!

Predicates & Quantifiers

Predicates & Quantifiers Review

- **Predicate**: a function that outputs true or false
  - Cat(x) := "x is a cat"
  - LessThan(x, y) := "$x < y$"

- **Domain of Discourse**: the types of inputs allowed in predicates
  - Numbers, mammals, cats and dogs, people in this class, etc.

- **Quantifiers**
  - Universal Quantifier $\forall x$: for all x, for every x
  - Existential Quantifier $\exists x$: there is an x, there exists an x, for some x

- **Domain Restriction**
  - Universal Quantifier ∀x: add the restriction as the hypothesis to an **implication**
  - Existential Quantifier $\exists x$: **AND** in the restriction

Problem 2 – ctrl-z

Translate these logical expressions to English. For each of the translations, assume that domain restriction is being used and take that into account in your English versions.

Let your domain be all UW Students. Predicates $143\text{Student}(x)$ and $311\text{Student}(x)$ mean the student is in CSE 143 and 311, respectively. $\text{BioMajor}(x)$ means x is a bio major, $\text{DidHomeworkOne}(x)$ means the student did homework 1 (of 311). Finally, $\text{KnowsJava}(x)$ and $\text{KnowsDeMorgan}(x)$ mean x knows Java and knows DeMorgan's Laws, respectively.

a) $\forall x\big(143\text{Student}(x) \rightarrow \text{KnowsJava}(x)\big)$

b) $\exists x\big(143\text{Student}(x) \wedge \text{BioMajor}(x)\big)$

c) $\forall x([311\text{Student}(x) \wedge \text{DidHomeworkOne}(x)] \rightarrow \text{KnowsDeMorgan}(x))$

Work on parts (a) and (c) with the people around you, and then we'll go over it together!

Problem 2 – ctrl-z

a) $\forall x(\text{143Student}(x) \rightarrow \text{KnowsJava}(x))$

Problem 2 – ctrl-z

a) $\forall x(\text{143Student}(x) \rightarrow \text{KnowsJava}(x))$

Every 143 student knows java.

"If a UW student is a 143 student, then they know java" is a valid translation of the original sentence, but it is not taking advantage of the domain restriction.

Problem 2 – ctrl-z

c) $\forall x([311\text{Student}(x) \land \text{DidHomeworkOne}(x)] \to \text{KnowsDeMorgan}(x))$

Problem 2 – ctrl-z

c) $\forall x([311\text{Student}(x) \land \text{DidHomeworkOne}(x)] \rightarrow \text{KnowsDeMorgan}(x))$

All 311 students who do Homework 1 know DeMorgan’s Laws.

“If a UW student is a 311 student and they did Homework 1, then they know DeMorgan’s Laws” is a valid translation of the original sentence, but it is not taking advantage of the domain restriction.

Problem 1 – Domain Restriction

Translate each of the following sentences into logical notation. These translations require some of our quantifier tricks. You may use the operators + and · which take two numbers as input and evaluate to their sum or product, respectively.

a) Domain: Positive integers; Predicates: *Even, Prime, Equal*
"There is only one positive integer that is prime and even."

b) Domain: Real numbers; Predicates: *Even, Prime, Equal*
"There are two different prime numbers that sum to an even number."

c) Domain: Real numbers; Predicates: *Even, Prime, Equal*
"The product of two distinct prime numbers is not prime."

d) Domain: Real numbers; Predicates: *Even, Prime, Equal, Positive, Greater, Integer*
"For every positive integer, there is a greater even integer"

Work on parts (a) and (b) with the people around you, and then we'll go over it together!

Problem 1 – Domain Restriction

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We can start out with:

$$\exists x(\mathrm{Prime}(x) \land \mathrm{Even}(x))$$

Problem 1 – Domain Restriction

a) Domain: Positive integers; Predicates: *Even, Prime, Equal*
"There is only one positive integer that is prime and even."

We can start out with:

$$\exists x(\text{Prime}(x) \land \text{Even}(x))$$

But now we need to add in the restriction that this x is the ONLY positive integer that is prime and even. This is a technique you'll use whenever you need to have only one of something:

$$\exists x(\text{Prime}(x) \land \text{Even}(x) \land \forall y[\neg\text{Equal}(x, y) \rightarrow \neg(\text{Even}(y) \land \text{Prime}(y))])$$

Or, we could use the contrapositive:

$$\exists x(\text{Prime}(x) \land \text{Even}(x) \land \forall y[(\text{Even}(y) \land \text{Prime}(y) \rightarrow \text{Equal}(x, y)])$$

Problem 1 – Domain Restriction

b) Domain: Real numbers; Predicates: *Even, Prime, Equal*
"There are two different prime numbers that sum to an even number."

Problem 1 – Domain Restriction

b) Domain: Real numbers; Predicates: *Even, Prime, Equal*
"There are two different prime numbers that sum to an even number."

Seems like maybe we should be able to say something like:

$$\exists x \exists y (\text{Prime}(x) \land \text{Prime}(y) \land \text{Even}(x + y))$$

Problem 1 – Domain Restriction

b) Domain: Real numbers; Predicates: *Even, Prime, Equal*
"There are two **different** prime numbers that sum to an even number."

Seems like maybe we should be able to say something like:

$$\exists x \exists y(\text{Prime}(x) \land \text{Prime}(y) \land \text{Even}(x+y))$$

But this leaves open the possibility of x and y being equal (so they won't be two DIFFERENT numbers). So, we need to explicitly add in that x and y are not equal:

$$\exists x \exists y(\text{Prime}(x) \land \text{Prime}(y) \land \text{Even}(x+y) \land \neg\text{Equal}(x,y))$$

Domain Restricting with Implications

Problem 9 There exists an Implication

Consider the expression $\exists x(P(x) \rightarrow Q(x))$

a) Suppose P(x) is <u>not always true</u> (so there exists an x such that P(x) is false). Explain why $\exists x(P(x) \rightarrow Q(x))$ holds **true**

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Consider the expression $\exists x(P(x) \rightarrow Q(x))$

a) Suppose P(x) is not always true (so there exists an x such that P(x) is false). Explain why $\exists x(P(x) \rightarrow Q(x))$ holds **true**

P(x)	Q(x)	P(x) → Q(x)
T	T	T
T	F	F
F	T	T
F	F	T

Suppose for some y such that P(y) is **false**

Problem 9 There exists an Implication

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a) Suppose P(x) is <u>not always true</u> (so there exists an x such that P(x) is false). Explain why $\exists x(P(x) \rightarrow Q(x))$ holds **true**

P(**y**)	Q(**y**)	P(**y**) → Q(**y**)
T	T	T
T	F	F
F	T	T
F	F	T

Suppose for some y such that P(y) is **false**

Plugging in y for x, we see that we get a **vacuous truth**

Problem 9 There exists an Implication

Consider the expression ∃x(P(x) → Q(x))

a) Suppose P(x) is <u>not always true</u> (so there exists an x such that P(x) is false). Explain why **∃x(P(x) → Q(x))** holds **true**

P(**y**)	Q(**y**)	P(**y**) → Q(**y**)
T	T	T
T	F	F
F	T	T
F	F	T

Suppose for some y such that P(y) is **false**

Plugging in y for x, we see that we get a **vacuous truth**

SO we have found a **single y** to make the whole expression "**∃x(P(x) → Q(x))**" hold **true**

Problem 9 There exists an Implication

Consider the expression $\exists x(P(x) \rightarrow Q(x))$

b) Now suppose $P(x)$ is <u>always true</u> (All values x allow $P(x)$ to be true)
Can we simplify the expression?

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b) Now suppose P(x) is <u>always true</u> (All values x allow P(x) to be true)
Can we simplify the expression?

P(x)	Q(x)	P(x) → Q(x)
T	T	T
T	F	F
F	T	T
F	F	T

Let's look at the same truth table…

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$P(x)$	$Q(x)$	$P(x) \rightarrow Q(x)$
T	T	T
T	F	F

Let's look at the same truth table…

But simplify the truth table since
P(x) is always true

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T	F	F

Let's look at the same truth table…

But simplify the truth table since
$P(x)$ is always true

Do you notice anything?

Problem 9 There exists an Implication

Consider the expression $\exists x(P(x) \rightarrow Q(x))$

b) Now suppose P(x) is <u>always true</u> (All values x allow P(x) to be true)
Can we simplify the expression?

$Q(x)$	$P(x) \rightarrow Q(x)$
T	T
F	F

Let's look at the same truth table…

But simplify the truth table since
P(x) is always true

Do you notice anything?

The **truth value** of **Q(x)** determines the truth value of **$P(x) \rightarrow Q(x)$**

Problem 9 There exists an Implication

Consider the expression $\exists x(P(x) \rightarrow Q(x))$

b) Now suppose P(x) is <u>always true</u> (All values x allow P(x) to be true)
Can we simplify the expression?

$Q(x)$	$P(x) \rightarrow Q(x)$
T	T
F	F

SO we can **simplify $\exists x(P(x) \rightarrow Q(x))$ to:**
$\exists x(Q(x))$ where P(x) is <u>always true</u>

How cool!!!

Problem 9 There exists an Implication

Consider the expression $\exists x(P(x) \rightarrow Q(x))$

(a) When $P(x)$ is sometimes true/true for some x (**∃x**): we get a vacuous **truth**

(b) When $P(x)$ is always true/true for all x (**∀x**): we can get **$\exists x Q(x)$**

Problem 9 There exists an Implication

Consider the expression $\exists x(P(x) \rightarrow Q(x))$

(a) When P(x) is sometimes true/true for some x (**∃x**): we get a vacuous **truth**

(b) When P(x) is always true/true for all x (**∀x**): we can get **∃xQ(x)**

Domain restricting for ∃x where x **can sometimes** be true, means the whole statement becomes **vacuously true.**

$\exists x(P(x) \rightarrow Q(x))$ becomes T

Takeaway: We would want $\exists x(P(x) \land Q(x))$ instead

This is why we only domain restrict with an implication for universal quantifiers ∀

$\forall x(P(x) \rightarrow Q(x))$

Problem 3 – Quantifier Switch

Consider the following pairs of sentences. For each pair, determine if one implies the other, if they are equivalent, or neither.

a) $\forall x\, \forall y\, P(x, y)$ $\qquad$ $\forall y\, \forall x\, P(x, y)$

b) $\exists x\, \exists y\, P(x, y)$ $\qquad$ $\exists y\, \exists x\, P(x, y)$

c) $\forall x\, \exists y\, P(x, y)$ $\qquad$ $\forall y\, \exists x\, P(x, y)$

d) $\forall x\, \exists y\, P(x, y)$ $\qquad$ $\exists x\, \forall y\, P(x, y)$

e) $\forall x\, \exists y\, P(x, y)$ $\qquad$ $\exists y\, \forall x\, P(x, y)$

Work on parts (d) and (e) with the people around you, and then we'll go over it together!

Problem 3 – Quantifier Switch

d) $\forall x \, \exists y \, P(x, y)$ $\qquad\qquad$ $\exists x \, \forall y \, P(x, y)$

Problem 3 – Quantifier Switch

d) $\forall x\, \exists y\, P(x,y)$ $\exists x\, \forall y\, P(x,y)$

Different!

For all x, there is a y **vs** there exists an x, that, for all y

Let P(x,y) be person x owns dog y

“All people own a dog”

	Robbie	Aruna	Anna	Jacob
				
				
				
				

VS

“There is person that owns all dogs”

	Robbie	Aruna	Anna	Jacob
				
				
				
				

Problem 3 – Quantifier Switch

d) $\forall x \, \exists y \, P(x, y)$ $\exists x \, \forall y \, P(x, y)$

Different!
For all x, there is a y **vs** there exists an x, that, for all y

Let P(x,y) be person x owns dog y

"All people own a dog"

	Robbie	Aruna	Anna	Jacob
	X			
				X
		X		
			X	

VS

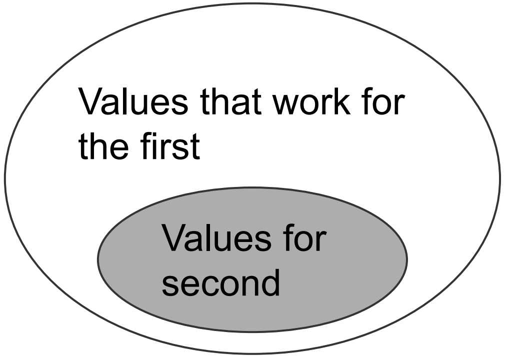
"There is person that owns all dogs"

	Robbie	Aruna	Anna	Jacob
	X			
	X			
	X			
	X			

Problem 3 – Quantifier Switch

e) $\forall x \, \exists y \, P(x, y)$ $\exists y \, \forall x \, P(x, y)$

Problem 3 – Quantifier Switch



e) $\forall x\, \exists y\, P(x, y)$ $\exists y\, \forall x\, P(x, y)$

The second implies the first

For all x, there is a y **vs** there exists a y, that, for all x

The second is **stronger** since a **specific y** must work **for all x** whereas for the first, the y value **does not** have to be the same **for every x**

"All people own a dog"

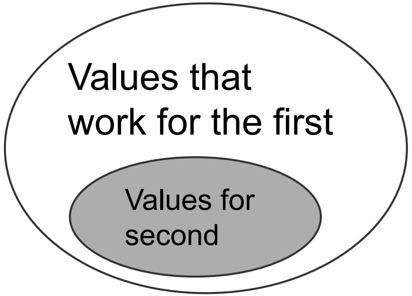
	Robbie	Aruna	Anna	Jacob
				
				
				
				

VS

"There is a dog owned by all people"

	Robbie	Aruna	Anna	Jacob
				
				
				
				

Problem 3 – Quantifier Switch



e) $\forall x \, \exists y \, P(x,y)$ $\exists y \, \forall x \, P(x,y)$

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"All people own a dog"

	Robbie	Aruna	Anna	Jacob
	X			
				X
		X		
			X	

VS

"There is a dog owned by all people"

	Robbie	Aruna	Anna	Jacob
				
				
	X	X	X	X
				

Direct Proofs

How can we prove an Implication?

We use a **direct proof** technique!

To prove: $\forall(P(x) \rightarrow Q(x))$

1. **Start:** Pretend/Assume that $P(x)$ is true
2. list some qualities of $P(x)$....
3. derive a new fact
4. that shows $Q(x)$
5. **End:** $Q(x)$

$P(x)$	$Q(x)$	$P(x) \rightarrow Q(x)$
T	T	T
T	F	F
F	T	T
F	F	T

Right now we will write these out in English and later formalize it with symbolic proofs!

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We use a **direct proof** technique!

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1. **Start:** Pretend/Assume that $P(x)$ is true
2. list some qualities of $P(x)$....
3. derive a new fact
4. that shows $Q(x)$
5. **End:** $Q(x)$

arbitrary x

…Since x was **arbitrary**

Right now we will write these out in English and later formalize it with symbolic proofs!

Writing a Proof (symbolically or in English)

- Don't just jump right in!
- Look at the **claim**, and make sure you know:
    - What every word in the claim means
    - What the claim as a whole means
- Translate the claim in predicate logic.
- Next, write down the **Proof Skeleton**:
    - Where to start
    - What your target is
- Then once you know what claim you are proving and your starting point and ending point, you can finally write the proof!

Helpful Tips for English Proofs

- Start by introducing your assumptions
  - Introduce variables with "let"
    - "Let x be an arbitrary prime number..."
  - Introduce assumptions with "suppose"
    - "Suppose that $y \in A \wedge y \notin B$..."

- When you supply a value for an existence proof, use "Consider"
  - "Consider $x = 2$..."

- **ALWAYS** state what type your variable is (integer, set, etc.)

- Universal Quantifier means variable must be arbitrary

- Existential Quantifier means variable can be specific

Problem 6 – Direct Proof

a) Let the domain of discourse be integers. Define the predicates $\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$, and $\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$. Translate the following claim into predicate logic:

The sum of an even and odd integer is odd.

b) Prove that the claim holds.

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Work on part (a) of this problem with the people around you, and then we'll go over it together!

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a) Let the domain of discourse be integers. Define the predicates $\mathrm{Odd}(x) := \exists k(x = 2k + 1)$, and $\mathrm{Even}(x) := \exists k(x = 2k)$. Translate the following claim into predicate logic:

The sum of an even and odd integer is odd.

$$\forall n \forall m\big(\big(\mathrm{Even}(n) \land \mathrm{Odd}(m)\big) \to \mathrm{Odd}(n + m)\big)$$

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The sum of an even and odd integer is odd.

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Lets walk through part (b) together!

Problem 6 – Direct Proof

$\mathrm{Odd}(x) := \exists k(x = 2k + 1)$
$\mathrm{Even}(x) := \exists k(x = 2k)$
Claim:
$\forall n \forall m\big(\big(\mathrm{Even}(n) \land \mathrm{Odd}(m)\big) \to \mathrm{Odd}(n + m)\big)$

b) Prove that the claim holds.

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$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n \forall m\big((\mathrm{Even}(n) \land \mathrm{Odd}(m)) \to \mathrm{Odd}(n + m)\big)$

b) Prove that the claim holds.

Let n and m be arbitrary integers.

…

Since n and m were arbitrary, the sum of any even and odd integer is odd.

Problem 6 – Direct Proof

$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n \forall m((\mathrm{Even}(n) \land \mathrm{Odd}(m)) \to \mathrm{Odd}(n + m))$

b) Prove that the claim holds.

Let n and m be arbitrary integers. Suppose n is even and m is odd.

…

Thus by (some reasoning here), $n + m$ is odd. Since n and m were arbitrary, the sum of any even and odd integer is odd.

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$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\text{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n \forall m\big((\text{Even}(n) \land \text{Odd}(m)\big) \rightarrow \text{Odd}(n + m))$

b) Prove that the claim holds.

Let n and m be arbitrary integers. Suppose n is even and m is odd. Then by definition of even, $n = 2k$ for some integer k. By definition of odd, $m = 2j + 1$ for some integer j.

…

Then $n + m$ is 2 times an integer plus 1. Thus by definition of odd, $n + m$ is odd.
Since n and m were arbitrary, the sum of any even and odd integer is odd.

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$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n \forall m\big((\mathrm{Even}(n) \wedge \mathrm{Odd}(m)\big) \rightarrow \mathrm{Odd}(n + m))$

b) Prove that the claim holds.

Let n and m be arbitrary integers. Suppose n is even and m is odd. Then by definition of even, $n = 2k$ for some integer k. By definition of odd, $m = 2j + 1$ for some integer j.

Then consider $n + m$:

$$n + m = 2k + 2j + 1$$

…

Then $n + m$ is 2 times an integer plus 1. Thus by definition of odd, $n + m$ is odd.
Since n and m were arbitrary, the sum of any even and odd integer is odd.

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$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
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b) Prove that the claim holds.

Let n and m be arbitrary integers. Suppose n is even and m is odd. Then by definition of even, $n = 2k$ for some integer k. By definition of odd, $m = 2j + 1$ for some integer j.

Then consider $n + m$:

$$\begin{aligned} n + m &= 2k + 2j + 1 \\ &= 2(k + j) + 1 \end{aligned}$$

…

Then $n + m$ is 2 times an integer plus 1. Thus by definition of odd, $n + m$ is odd.
Since n and m were arbitrary, the sum of any even and odd integer is odd.

Problem 6 – Direct Proof

$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
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Claim:
$\forall n \forall m\big((\mathrm{Even}(n) \land \mathrm{Odd}(m)\big) \rightarrow \mathrm{Odd}(n + m))$

b) Prove that the claim holds.

Let n and m be arbitrary integers. Suppose n is even and m is odd. Then by definition of even, $n = 2k$ for some integer k. By definition of odd, $m = 2j + 1$ for some integer j.

Then consider $n + m$:

$$\begin{aligned} n + m &= 2k + 2j + 1 \\ &= 2(k + j) + 1 \end{aligned}$$

Since k and j are integers, $k + j$ is an integer.
Then $n + m$ is 2 times an integer plus 1. Thus by definition of odd, $n + m$ is odd.
Since n and m were arbitrary, the sum of any even and odd integer is odd.

Biconditionals (bonus)

Problem 7 – Proof of Biconditional

a) Let the domain of discourse be integers. Define the predicates $\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$, and $\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$. Translate the following claim into predicate logic:

For all integers n, $n - 4$ is even if and only if $n + 17$ is odd.

b) Prove that the claim holds.

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For all integers n, $n - 4$ is even if and only if $n + 17$ is odd.

Work on part (a) of this problem with the people around you, and then we'll go over it together!

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For all integers n, $n - 4$ is even if and only if $n + 17$ is odd.

$\forall n(\text{Even}(n - 4) \leftrightarrow \text{Odd}(n + 17))$

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For all integers n, $n - 4$ is even if and only if $n + 17$ is odd.

b) Prove that the claim holds.

Lets walk through part (b) together!

Problem 7 – Proof of Biconditional

$$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$$
$$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$$

Claim:

$$\forall n(\mathrm{Even}(n - 4) \leftrightarrow \mathrm{Odd}(n + 17))$$

b) Prove that the claim holds.

Problem 7 – Proof of Biconditional

$$\text{Odd}(x) := \exists k(x = 2k + 1)$$
$$\text{Even}(x) := \exists k(x = 2k)$$

Claim:

$$\forall n(\text{Even}(n-4) \leftrightarrow \text{Odd}(n+17))$$

b) Prove that the claim holds.

We know that a biconditional $p \leftrightarrow q$ can be equivalently expressed as two implications anded together: $p \rightarrow q \land q \rightarrow p$. So, in order to prove a biconditional, we need to prove both implications hold.

For this problem, we need to prove both the forward direction:

$$\forall n(\text{Even}(n-4) \rightarrow \text{Odd}(n+17))$$

And the backward direction:

$$\forall n(\text{Odd}(n+17) \rightarrow \text{Even}(n-4))$$

By showing both implications hold, we prove that the biconditional holds.

Problem 7 – Proof of Biconditional

$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\text{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n(\text{Even}(n-4) \leftrightarrow \text{Odd}(n+17))$

b) Prove that the claim holds.

$\Rightarrow$ Let n be an arbitrary integer.

…

Since n was arbitrary, we have shown that for all integers n that if $n - 4$ is even, then $n + 17$ is odd.

Problem 7 – Proof of Biconditional

$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n(\mathrm{Even}(n-4) \leftrightarrow \mathrm{Odd}(n+17))$

b) Prove that the claim holds.

$\Rightarrow$ Let n be an arbitrary integer. Suppose that $n-4$ is even. Then by definition of even, $n - 4 = 2k$ for some integer k. Then observe that:

$n - 4 = 2k$	
$n + 17 = 2k + 21$	Adding 21 to both sides
$n + 17 = 2(k + 10) + 1$	Factoring

Thus $n + 17 = 2(k + 10) + 1$.
Since k is an integer, $k + 10$ is an integer. So $n + 17$ is 2 times an integer plus 1. Thus by definition of odd, $n + 17$ is odd.
Since n was arbitrary, we have shown that for all integers n that if $n - 4$ is even, then $n + 17$ is odd.

Problem 7 – Proof of Biconditional

$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n(\mathrm{Even}(n - 4) \leftrightarrow \mathrm{Odd}(n + 17))$

b) Prove that the claim holds.

$\Leftarrow$ Let n be an arbitrary integer.

…

Since n was arbitrary, we have shown that for all integers n, if $n + 17$ is odd, then $n - 4$ is even.

Problem 7 – Proof of Biconditional

$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n(\mathrm{Even}(n-4) \leftrightarrow \mathrm{Odd}(n+17))$

b) Prove that the claim holds.

$\Leftarrow$ Let n be an arbitrary integer. Suppose $n + 17$ is odd. Then by definition of odd, $n + 17 = 2k + 1$ for some integer k. Then observe that:

$$
\begin{aligned}
n + 17 &= 2k + 1 \\
n - 4 &= 2k + 1 - 21 && \text{Subtracting 21 from both sides} \\
n - 4 &= 2(k - 10) && \text{Factoring}
\end{aligned}
$$

Thus $n - 4 = 2(k - 10)$.
Since k is an integer, $k - 10$ is an integer. So $n - 4$ is 2 times an integer.
So by definition of even, $n - 4$ is even. Since n was arbitrary, we have shown that for all integers n, if $n + 17$ is odd, then $n - 4$ is even.

Problem 7 – Proof of Biconditional

$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\text{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n(\text{Even}(n-4) \leftrightarrow \text{Odd}(n+17))$

b) Prove that the claim holds.

$\Rightarrow$ Let n be an arbitrary integer. Suppose that $n - 4$ is even. Then by definition of even, $n - 4 = 2k$ for some integer k. Then observe that:

$n - 4 = 2k$	
$n + 17 = 2k + 21$	Adding 21 to both sides
$n + 17 = 2(k + 10) + 1$	Factoring

Thus $n + 17 = 2(k + 10) + 1$. Since k is an integer, $k + 10$ is an integer. So $n + 17$ is 2 times an integer plus 1. Thus by definition of odd, $n + 17$ is odd. Since n was arbitrary, we have shown that for all integers n that if $n - 4$ is even, then $n + 17$ is odd.

$\Leftarrow$ Let n be an arbitrary integer. Suppose $n + 17$ is odd. Then by definition of odd, $n + 17 = 2k + 1$ for some integer k. Then observe that:

$n + 17 = 2k + 1$	
$n - 4 = 2k + 1 - 21$	Subtracting 21 from both sides
$n - 4 = 2(k - 10)$	Factoring

Thus $n - 4 = 2(k - 10)$. Since k is an integer, $k - 10$ is an integer. So $n - 4$ is 2 times an integer. So by definition of even, $n - 4$ is even. Since n was arbitrary, we have shown that for all integers n, if $n + 17$ is odd, then $n - 4$ is even.

That’s All, Folks!

Thanks for coming to section this week!
Any questions?