Section MR: Midterm Review

1. Midterm Review: Translation

Let your domain of discourse be all coffee drinks. You should use the following predicates:

- soy(x) is true iff x contains soy milk.
- whole(x) is true iff x contains whole milk.
- sugar(x) is true iff x contains sugar
- decaf(x) is true iff x is not caffeinated.
- vegan(x) is true iff x is vegan.
- RobbieLikes(x) is true iff Robbie likes the drink x.

Translate each of the following statements into predicate logic. You may use quantifiers, the predicates above, and usual math connectors like $=$ and $\neq$.

(a) Coffee drinks with whole milk are not vegan.

(b) Robbie only likes one coffee drink, and that drink is not vegan.

(c) There is a drink that has both sugar and soy milk.

(d) Translate the following symbolic logic statement into a (natural) English sentence. Take advantage of domain restriction.

$$\forall x([\texttt{decaf}(x) \land \texttt{RobbieLikes}(x)] \rightarrow \texttt{sugar}(x))$$

2. Midterm Review: Even Steven

Prove that for all integers k, k(k +3) is even.

(a) Let your domain be integers. Write the predicate logic of this claim. Define predicates Odd and Even!

(b) Write an English proof for this claim.

3. Midterm Review: Number Theory

Let p be a prime number at least 3, and let x be an integer such that $x^2\%p = 1$.

(a) Show that if an integer y satisfies $y \equiv 1 \pmod{p}$, then $y^2 \equiv 1 \pmod{p}$. (this proof will be short!) (Try to do this without using the theorem "Raising Congruences To A Power")

(b) Repeat part (a), but don't use any theorems from the Number Theory Reference Sheet. That is, show the claim directly from the definitions.

(c) From part (a), we can see that $x\%p$ can equal 1. Show that for any integer x, if $x^2 \equiv 1 \pmod{p}$, then $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$. That is, show that the only value $x\%p$ can take other than 1 is $p - 1$.

Hint: Suppose you have an x such that $x^2 \equiv 1 \pmod{p}$ and use the fact that $x^2 - 1 = (x-1)(x+1)$
Hint: You may the following theorem without proof: if p is prime and $p \mid (ab)$ then $p \mid a$ or $p \mid b$.

4. Midterm Review: Induction

For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = 1^2 + 2^2 + \cdots + n^2.$$

Prove that for all $n \in \mathbb{N}$, $S_n = \frac{1}{6}n(n+1)(2n+1)$.

5. Midterm Review: Strong Induction

Robbie is planning to buy snacks for the members of his competitive roller-skating troupe. However, his local grocery store sells snacks in packs of 5 and packs of 7.

Prove that Robbie can buy exactly n snacks for all integers $n \geq 24$

Proof By Contradiction, Set Theory Practice

6. Wait, That Doesn't Add Up

Write a proof by contradiction for the following proposition: There exist no integers x and y such that $18x + 6y = 1$. In predicate logic this could be expressed as $\forall x \forall y (18x + 6y \neq 1)$. HINT: Try negating this statement before writing your proof.

7. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

(b) $B = \{\mathbf{\{\}},\ \{\mathbf{\{\}}\},\ \{\mathbf{\{\}}, \mathbf{\{\}}\},\ \{\mathbf{\{\}}, \mathbf{\{\}}, \mathbf{\{\}}\},\ \ldots\}$

(c) $C = A \times (B \cup \{7\})$

(d) $D = \varnothing$

(e) $E = \{\varnothing\}$

(f) $F = \mathcal{P}(\{\varnothing\})$

8. Set = Set

Prove the following set identities. Write both a formal inference proof **and** an English proof.

(a) Let the universal set be $\mathcal{U}$. Prove $A \cap \overline{B} \subseteq A \backslash B$ for any sets A, B.

(b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

9. Set Equality

(a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.