Section 07: Structural Induction & Set Theory

1. Just The Setup

For each of these statements,

- Translate the sentence into predicate logic.
- Write the first few and last few steps of an inference proof of the statement (you do not need to write the middle – just enough to introduce all givens and assumptions and the conclusion at the end)
- Write the first few sentences and last few sentences of the English proof.

(a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ for any sets A, B, C.

2. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

(b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$

(c) $C = A \times (B \cup \{7\})$

(d) $D = \varnothing$

(e) $E = \{\varnothing\}$

(f) $F = \mathcal{P}(\{\varnothing\})$

3. Set = Set

Prove the following set identities. Write both a formal inference proof **and** an English proof.

(a) Let the universal set be $\mathcal{U}$. Prove $A \cap \overline{B} \subseteq A \backslash B$ for any sets A, B.

(b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

4. Set Equality

(a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

(b) Let $\mathcal{U}$ be the universal set. Show that $\overline{\overline{X}} = X$.

5. Structural Induction

(a) Consider the following recursive definition of strings.

Basis Step: `""` is a string

Recursive Step: If X is a string and c is a character then $\mathsf{append}(c, X)$ is a string.

Recall the following recursive definition of the function len:

$$\begin{aligned} \mathsf{len}(\texttt{""}) &= 0 \\ \mathsf{len}(\mathsf{append}(c, X)) &= 1 + \mathsf{len}(X) \end{aligned}$$

Now, consider the following recursive definition:

$$\begin{aligned} \mathsf{double}(\texttt{""}) &= \texttt{""} \\ \mathsf{double}(\mathsf{append}(c, X)) &= \mathsf{append}(c, \mathsf{append}(c, \mathsf{double}(X))). \end{aligned}$$

Prove that for any string X, $\mathsf{len}(\mathsf{double}(X)) = 2\mathsf{len}(X)$.

(b) Consider the following definition of a (binary) **Tree**:

Basis Step: $\bullet$ is a **Tree**.

Recursive Step: If L is a **Tree** and R is a **Tree** then $\texttt{Tree}(\bullet, L, R)$ is a **Tree**.

The function leaves returns the number of leaves of a **Tree**. It is defined as follows:

$$\begin{aligned} \mathsf{leaves}(\bullet) &= 1 \\ \mathsf{leaves}(\texttt{Tree}(\bullet, L, R)) &= \mathsf{leaves}(L) + \mathsf{leaves}(R) \end{aligned}$$

Also, recall the definition of size on trees:

$$\begin{aligned} \mathsf{size}(\bullet) &= 1 \\ \mathsf{size}(\texttt{Tree}(\bullet, L, R)) &= 1 + \mathsf{size}(L) + \mathsf{size}(R) \end{aligned}$$

Prove that $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$ for all Trees T.

(c) Prove the previous claim using strong induction. Define $P(n)$ as "all trees T of size n satisfy $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$". You may use the following facts:

- For any tree T we have $\mathsf{size}(T) \geq 1$.
- For any tree T, $\mathsf{size}(T) = 1$ if and only if $T = \bullet$.

If we wanted to prove these claims, we could do so by structural induction.

Note, in the inductive step you should start by letting T be an arbitrary tree of size $k + 1$.

6. Reversing a Binary Tree

Consider the following definition of a (binary) **Tree**.

Basis Step $\texttt{Nil}$ is a **Tree**.

Recursive Step If L is a **Tree**, R is a **Tree**, and x is an integer, then $\texttt{Tree}(x, L, R)$ is a **Tree**.

The sum function returns the sum of all elements in a **Tree**.

$$\begin{aligned} \mathsf{sum}(\texttt{Nil}) &= 0 \\ \mathsf{sum}(\texttt{Tree}(x, L, R)) &= x + \mathsf{sum}(L) + \mathsf{sum}(R) \end{aligned}$$

The following recursively defined function produces the mirror image of a **Tree**.

$$\begin{aligned} \mathsf{reverse}(\texttt{Nil}) &= \texttt{Nil} \\ \mathsf{reverse}(\texttt{Tree}(x, L, R)) &= \texttt{Tree}(x, \mathsf{reverse}(R), \mathsf{reverse}(L)) \end{aligned}$$

Show that, for all **Tree**s T that

$$\mathsf{sum}(T) = \mathsf{sum}(\mathsf{reverse}(T))$$

7. Walk the Dawgs

Suppose a dog walker takes care of $n \geq 12$ dogs. The dog walker is not a strong person, and will walk dogs in groups of 3 or 7 at a time (every dog gets walked exactly once). Prove the dog walker can always split the n dogs into groups of 3 or 7.

8. For All

For this problem, we'll see an incorrect use of induction. For this problem, we'll think of all of the following as binary trees:

- A single node.
- A root node, with a left child that is the root of a binary tree (and no right child)
- A root node, with a right child that is the root of a binary tree (and no left child)
- A root node, with both left and right children that are roots of binary trees.

Let $P(n)$ be "for all trees of height n, the tree has an odd number of nodes"

Take a moment to realize this claim is false.

Now let's see an incorrect proof:

We'll prove $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$): There is only one tree of height 0, a single node. It has one node, and $1 = 2 \cdot 0 + 1$, which is an odd number of nodes.

Inductive Hypothesis: Suppose $P(i)$ holds for $i = 0, ..., k$, for some arbitrary $k \geq 0$.

Inductive Step: Let T be an arbitrary tree of height k. All trees with nodes (and since $k \geq 0$, T has at least one node) have a leaf node. Add a left child and right child to a leaf (pick arbitrarily if there's more than one), This tree now has height $k + 1$ (since T was height k and we added children below). By IH, T had an odd number of nodes, call it $2j + 1$ for some integer j. Now we have added two more, so our new tree has $2j + 1 + 2 = 2(j + 1) + 1$ nodes. Since j was an integer, so is $j + 1$, and our new tree has an odd number of nodes, as required, so $P(k + 1)$ holds.

By the principle of induction, $P(n)$ holds for all $n \in \mathbb{N}$. Since every tree has an (integer) height of 0 or more, every tree is included in some $P()$, so the claim holds for all trees.

(a) What is the bug in the proof?

(b) What should the starting point and target of the IS be (you can't write a full proof, as the claim is false).

9. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.