Section 10: Irregularity, Cardinality & Uncomputability

1. Irregularity

(a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n 1^n 0^n \ : \ n \geq 0\}$ is not regular.

(b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n (12)^m \ : \ n \geq m \geq 0\}$ is not regular.

2. Cardinality

(a) You are a pirate. You begin in a square on a 2D grid which is infinite in all directions. In other words, wherever you are, you may move up, down, left, or right. Some single square on the infinite grid has treasure on it. Find a way to ensure you find the treasure in finitely many moves.

(b) Prove that $\{3x \ : \ x \in \mathbb{N}\}$ is countable.

(c) Prove that the set of irrational numbers is uncountable.
Hint: Use the fact that the rationals are countable and that the reals are uncountable.

(d) Prove that $\mathcal{P}(\mathbb{N})$ is uncountable.

3. Countable Unions

(a) Show that $\mathbb{N} \times \mathbb{N}$ is countable.

Hint: How did we show the rationals were countable?

(b) Show that the countable union of countable sets is countable. That is, given a collection of sets $S_1, S_2, S_2, \ldots$ such that S_i is countable for all $i \in \mathbb{N}$, show that

$$S = S_1 \cup S_2 \cup \cdots = \{x : x \in S_i \text{ for some } i\}$$

is countable.

Hint: Find a way labeling the elements and see if you can apply the previous part to construct an onto function from $\mathbb{N}$ to S.

4. Uncomputability

(a) Let $\Sigma = \{0, 1\}$. Prove that the set of palindromes is decidable.

(b) Prove that the set $\{(\texttt{CODE}(R), x, y) \;\; : \;\; R$ is a program and $R(x) \neq R(y)\}$ is undecidable where $R(x)$ is the output string that R produces on input x if R halts and we write $R(x) = \uparrow$ if R runs forever.

(a) Given a CFG, it is impossible to determine that it defines a regular language. (T/F)

(b) All uncountable sets have the same cardinality as the real numbers. (T/F)

(c) The set of all valid python programs has the same cardinality of the set of integers.(T/F)

5. Final Review: Translations

Translate the following sentences into logical notation if the English statement is given or to an English statement if the logical statement is given, taking into account the domain restriction.

Let the domain of discourse be students and courses.

Use predicates $\texttt{Student}, \texttt{Course}, \texttt{CseCourse}$ to do the domain restriction.

You can use $\texttt{Taking}(x, y)$ which is true if and only if x is taking y. You can also use $\texttt{RobbieTeaches}(x)$ if and only if Robbie teaches x and $\texttt{ContainsTheory}(x)$ if and only if x contains theory.

(a) Every student is taking some course.

(b) There is a student that is not taking every cse course.

(c) Some student has taken only one cse course.

(d) $\forall x[(\texttt{Course}(x) \wedge \texttt{RobbieTeaches}(x)) \rightarrow \texttt{ContainsTheory}(x)]$

(e) $\exists x\, \texttt{CseCourse}(x) \wedge \texttt{RobbieTeaches}(x) \wedge \texttt{ContainsTheory}(x) \wedge \forall y((\texttt{CseCourse}(y) \wedge \texttt{RobbieTeaches}(y)) \rightarrow x = y)$

6. Review: Set Theory

Suppose that $A \subseteq B$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

7. Review: Functions

Let $f : X \rightarrow Y$ be a function. For a subset C of X, define $f(C)$ to be the set of elements that f sends C to. In other words, $f(C) = \{f(c) : c \in C\}$.

Let A, B be subsets of X. Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$.

8. Review: Induction

(a) A Husky Tree is a tree built by the following definition:

Basis: A single gold node is a Husky Tree.

Recursive Rules:

1. Let T_1, T_2 be two Husky Trees, both with root nodes colored gold. Make a new purple root node and attach the roots of T_1, T_2 to the new node to make a new Husky Tree.
2. Let T_1, T_2 be two Husky Trees, both with root nodes colored purple. Make a new purple root node and attach the roots of T_1, T_2 to the new node to make a new Husky Tree.
3. Let T_1, T_2 be two Husky Trees, one with a purple root, the other with a gold root. Make a new gold root node, and attatch the roots of T_1, T_2 to the new node to make a new Husky Tree.

Use structural induction to show that for every Husky Tree: if it has a purple root, then it has an even number of leaves and if it has a gold root, then it has an odd number of leaves.

(b) Use induction to prove that for every positive integer n,

$$1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$$

9. Review: Languages

(a) Construct a regular expression that represents binary strings where no occurrence of 11 is followed by a 0.

(b) Construct a CFG that represents the following language: $\{1^x 2^y 3^y 4^x : x, y \geq 0\}$

(c) Construct a DFA that recognizes the language of all binary strings which, when interpreted as a binary number, are divisible by 3. e.g. 11 is 3 in base-10, so should be accepted while 111 is 7 in base-10, so should be rejected. The first bit processed will be the **most**-significant bit.
Hint: you need to keep track of the remainder %3. What happens to a binary number when you add a 0 at the end? A 1? It's a lot like a shift operation...

(d) Construct a DFA that recognizes the language of all binary strings with an even number of 0's and each 0 is (immediately) followed by at least one 1.

10. Review: Uncountability

Let S be the set of all real numbers in $[0, 1)$ that only have 0s and 1s in their decimal representation. Prove that S is uncountably infinite.