Homework 1

Due in class on 15 April 2011

Counting

1. How many binary strings of length seven have:

    (a) three 0s and four 1s?

    (b) three or more 0s?

    (c) fewer than three 1s?

    (d) How many ternary (0, 1, 2) strings of length ten have three 0s, four 1s and three 2s?

    (e) How many nonary (0-8) strings of length seven exist where every digit is distinct? (i.e. each digit occurs at most once.)

    [15 points]

2. How many ways can you choose a dozen donuts from 20 varieties:

    (a) if there are no two donuts of the same variety?

    (b) if all donuts are of the same variety?

    (c) if there are no restrictions?

    (d) if there are at least two varieties?

    (e) if there must be at least six blueberry-filled donuts?

    (f) if you want six blueberry donuts, four chocolate donuts and two Boston creams?

    (g) How many ways are there to distribute a dozen donuts among three hungry computer scientists (ignoring donut variety)?

    This question was brought to you with limited commercial interruption by Top Pot Doughnuts (TM).

    [21 points]

3. (a) What is the coefficient of $x^{20}y^{30}$ in $(2x + 3y)^{50}$?

    (b) What is the coefficient of $x^{20}y^{30}z^{40}$ in $(2x + 3y + 4z)^{90}$?

    (Just write the expressions; you don't need to calculate the numerical values.)

    [6 points]

4. Let $(x_i, y_i), i \in \{1, 2, 3, 4, 5\}$ be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates. (Hint: Pigeonhole Principle.)

    [5 points]

5. In a class of 80 people, some are CS majors, some are Math majors, some are Math *minors* and some are Econ minors. (Some fall into more than one of these categories, and some fall into none.)

   - Nobody is both a Math major and a Math minor.
   - Nobody has more than one minor.
   - 40 are CS majors.
   - 15 are Math majors.
   - 20 are Math minors.
   - 18 are Econ minors.
   - 7 are both Math and CS Majors.
   - 4 of the CS majors are Econ minors.
   - 11 of the CS majors are Math minors.
   - 5 of the math majors are Econ minors.
   - 13 people do not fit in any of these categories, i.e. they are not CS majors, Math majors, Math minors or Econ minors.

   How many people are CS-Math dual majors with Econ minors? (Hint: draw a Venn or Euler diagram.)

   [4 points]

Probability

6. For question 5 above, what is the probability that a random student:

   (a) is a CS major?
   (b) is an Econ minor but not a CS major?
   (c) is a CS-Math dual major with an Econ minor?
   (d) is an Econ minor, and either a CS major or a Math major (but not both)?

   [8 points]

7. Olympic gold medalist swimmer Cullen Jones was born on February 29. What is the probability that of 100 randomly chosen people, none of them share Jones' birthday? (Ignore the year. Assume that February 29 is 0.25 times as likely as any other date, and all other dates are uniformly distributed.)

   [3 points]

8. You perform a set of independent trials consisting of rolling a **pair** of fair dice and recording their sum. What is the probability of an outcome of 5 appearing before an outcome of 8?

   [5 points]

9. In preparation for a projected epidemic of narwhal flu, the FDA administers an experimental new vaccine to 80% of the population. The vaccine can either make you sick (with probability 20%) or give you immunity to the disease (probability 80%). The probability of a randomly chosen person from the whole population being infected is 24%.

   (a) What is the probability that a randomly chosen non-vaccinated person gets infected?

(b) For a randomly chosen sick person, what is the probability that he or she received the vaccine?

(c) If you wish to minimize your chances of getting sick, should you accept the vaccine? Why or why not?

[9 points]

10. Three psychic octopuses (octopi? octopodes?) predict soccer results. Each octopus is correct with a probability of $\frac{2}{3}$, and their predictions are independent.

(a) What is the probability that their majority vote is correct?

(b) If the majority vote is incorrect, what is the probability that each octopus is correct?

(c) If two octopuses are wrong, what is the probability that the third is correct?

(d) If two octopuses agree with each other, what is the probability that the third will agree with them?

[12 points]

11. Knaves always lie and knights always tell the truth. You are on an island whose population of 1000 is 60% knights and 40% knaves. You run into John and Bill, two randomly chosen inhabitants.

(a) What is the probability that John and Bill are both knaves?

(b) What is the probability that either John or Bill is a knave (or both)?

(c) Are the events that Bill and John are knaves independent?

(d) If John says, "we are both knaves", what are the conditional probabilities that John and Bill are knaves?

[12 points]