Whither the Binomial…

- Recall example of sending bit string over network
  - n = 4 bits sent over network where each bit had independent probability of corruption p = 0.1
  - X = number of bit corrupted. X ~ Bin(4, 0.1)
  - In real networks, send large bit strings (length $n \approx 10^4$)
  - Probability of bit corruption is very small $p \approx 10^{-6}$
  - X ~ Bin(10^4, 10^{-6}) is unwieldy to compute
- Extreme n and p values arise in many cases
  - # bit errors in file written to disk (# of typos in a book)
  - # of elements in particular bucket of large hash table
  - # of servers crashes in a day in giant data center
  - # Facebook login requests that go to particular server

Binomial in the Limit

- Recall the Binomial distribution

$$P(X=i)=\frac{n!}{i!(n-i)!}p^i(1-p)^{n-i}$$

- Let $\lambda = np$ (equivalently: $p = \lambda/n$)

$$P(X=i)=\frac{n!}{i!(n-i)!}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}=\frac{n(n-1)\ldots(n-i+1)}{n^i}\frac{\lambda^i}{i!}\frac{(1-\lambda/n)^n}{(1-\lambda/n)^i}$$

- When n is large, p is small, and λ is "moderate":

$$\frac{n(n-1)\ldots(n-i+1)}{n^i}\approx 1 \qquad (1-\lambda/n)^n \approx e^{-\lambda} \qquad (1-\lambda/n)^i \approx 1$$

- Yielding: $P(X=i)\approx 1\frac{\lambda^i}{i!}\frac{e^{-\lambda}}{1}=\frac{\lambda^i}{i!}e^{-\lambda}$

Poisson Random Variable

- X is a **<u>Poisson</u>** Random Variable: X ~ Poi(λ)
  - X takes on values 0, 1, 2…
  - and, for a given parameter $\lambda > 0$,
  - has distribution (PMF):

$$P(X=i)=e^{-\lambda}\frac{\lambda^i}{i!}$$

- Note Taylor series: $e^{\lambda}=\frac{\lambda^0}{0!}+\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\ldots=\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}$
- So: $\sum_{i=0}^{\infty}P(X=i)=\sum_{i=0}^{\infty}e^{-\lambda}\frac{\lambda^i}{i!}=e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}=e^{-\lambda}e^{\lambda}=1$

Sending Data on Network Redux

- Recall example of sending bit string over network
  - Send bit string of length $n = 10^4$
  - Probability of (independent) bit corruption $p = 10^{-6}$
  - X ~ Poi($\lambda = 10^4 * 10^{-6} = 0.01$)
  - What is probability that message arrives uncorrupted?

$$P(X=0)=e^{-\lambda}\frac{\lambda^i}{i!}=e^{-0.01}\frac{(0.01)^0}{0!}\approx 0.990049834$$

  - Using Y ~ Bin(10^4, 10^{-6}):

$$P(Y=0)\approx 0.990049829$$

Caveat emptor: Binomial computed with built-in function in R software package, so some approximation may have occurred. Approximation are closer to you than they may appear in some software packages.

Simeon-Denis Poisson

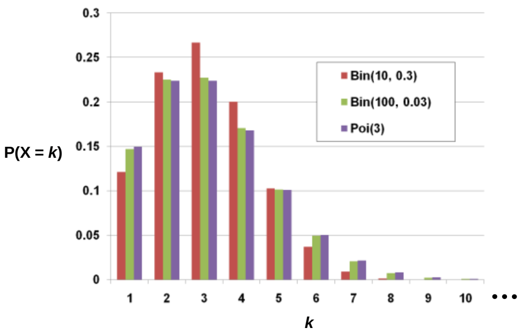
- Simeon-Denis Poisson (1781-1840) was a prolific French mathematician

- Published his first paper at 18, became professor at 21, and published over 300 papers in his life
  - He reportedly said *"Life is good for only two things, discovering mathematics and teaching mathematics."*
- Definitely did not look like Charlie Sheen

Poisson Random is Binomial in Limit

- Poisson approximates Binomial where n is large, p is small, and $\lambda = np$ is "moderate"
- Different interpretations of "moderate"
  - $n > 20$ and $p < 0.05$
  - $n > 100$ and $p < 0.1$
- Really, Poisson is Binomial as
  $n \rightarrow \infty$ and $p \rightarrow 0$, where $np = \lambda$

Bin(10, 0.3), Bin(100, 0.03) vs. Poi(3)



Tender (Central) Moments with Poisson

- Recall: Y ~ Bin(n, p)
  - E[Y] = np
  - Var(Y) = $np(1 – p)$
- X ~ Poi(λ) where $\lambda = np$ ($n \to \infty$ and $p \to 0$)
  - E[X] = $np = \lambda$
  - Var(X) = $np(1 – p) = \lambda(1 – 0) = \lambda$
  - Yes, expectation and variance of Poisson are same
    - It brings a tear to my eye…
  - Recall: Var(X) = E[X^2] – (E[X])^2
  - E[X^2] = Var(X) + (E[X])^2 = $\lambda + \lambda^2 = \lambda(1 + \lambda)$

It's Really All About Raisin Cake

- Bake a cake using *many* raisins and *lots* of batter
- Cake is enormous (in fact, infinitely so…)
  - Cut slices of "moderate" size (w.r.t. # raisins/slice)
  - Probability p that a particular raisin is in a certain slice is very small (p = 1/# cake slices)
- Let X = number of raisins in a certain cake slice
- X ~ Poi(λ), where $\lambda = \frac{\text{\# raisins}}{\text{\# cake slices}}$

CS = Baking Raisin Cake With Code

- Hash tables
  - strings = raisins
  - buckets = cake slices
- Server crashes in data center
  - servers = raisins
  - list of crashed machines = particular slice of cake
- Facebook login requests (i.e., web server requests)
  - requests = raisins
  - server receiving request = cake slice

Defective Chips

- Computer chips are produced
  - p = 0.1 that a chip is defective
  - Consider a sample of n = 10 chips
  - What is P(sample contains $\leq$ 1 defective chip)?
  - Using Y ~ Bin(10, 0.1):

$$P(Y \leq 1) = \binom{10}{0}(0.1)^0(1-0.1)^{10} + \binom{10}{1}(0.1)^1(1-0.1)^9 \approx 0.7361$$

  - Using X ~ Poi(λ = (0.1)(10) = 1)

$$P(X \leq 1) = e^{-1}\frac{1^0}{0!} + e^{-1}\frac{1^1}{1!} = 2e^{-1} \approx 0.7358$$

Efficiently Computing Poisson

- Let X ~ Poi(λ)
  - Want to compute P(X = i) for multiple values of i
  - E.g., Computing $P(X \leq a) = \sum_{i=0}^{a} P(X = i)$
- Iterative formulation:
  - Compute P(X = i + 1) from P(X = i)

$$\frac{P(X = i+1)}{P(X = i)} = \frac{e^{-\lambda}\lambda^{i+1}/(i+1)!}{e^{-\lambda}\lambda^i / i!} = \frac{\lambda}{i+1}$$

  - Use recurrence relation:

$$P(X = 0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-\lambda}$$

$$P(X = i+1) = \frac{\lambda}{i+1}P(X = i)$$

Approximately Poisson Approximation

- Poisson can still provide good approximation even when assumptions "mildly" violated
- "Poisson Paradigm"
- Can apply Poisson approximation when...
  - "Successes" in trials are not entirely independent
    - Example: # entries in each bucket in large hash table
  - Probability of "Success" in each trial varies (slightly)
    - Small relative change in a very small p
    - Example: average # requests to web server/sec. may fluctuate slightly due to load on network

Birthday Problem Redux

- What is the probability that of *n* people, none share the same birthday (regardless of year)?
  - $n = \binom{n}{2}$ trials, one for each pair of people (x, y), $x \neq y$
  - Let $E_{x,y}$ = x and y have same birthday (trial success)
  - $P(E_{x,y}) = p = 1/365$ (note: all $E_{x,y}$ not independent)
  - X ~ Poi(λ) where $\lambda = \binom{n}{2}\frac{1}{365} = \frac{n(n-1)}{730}$

$$P(X = 0) = e^{-n(n-1)/730}\frac{(n(n-1)/730)^0}{0!} = e^{-n(n-1)/730}$$

  - Solve for smallest integer *n*, s.t.: $e^{-n(n-1)/730} \leq 0.5$

$$\ln(e^{-n(n-1)/730}) \leq \ln(0.5) \rightarrow n(n-1) \geq 730\ln(0.5) \rightarrow n \geq 23$$

  - Same as before!

Poisson Processes

- Consider "rare" events that occur over time
  - Earthquakes, radioactive decay, hits to web server, etc.
  - Have time interval for events (1 year, 1 sec, whatever...)
  - Events arrive at rate: λ events per interval of time
- Split time interval into $n \rightarrow \infty$ sub-intervals
  - Assume at most one event per sub-interval
  - Event occurrences in sub-intervals are independent
  - With many sub-intervals, probability of event occurring in any given sub-interval is small
- N(t) = # events in original time interval ~ Poi(λ)

Web Server Load

- Consider requests to a web server in 1 second
  - In past, server load averages 2 hits/second
  - X = # hits server receives in a second
  - What is P(X = 5)?
- Model
  - Assume server cannot acknowledge > 1 hit/msec.
  - 1 sec = 1000 msec. (= large *n*)
  - P(hit server in 1 msec) = 2/1000 (= small *p*)
  - X ~ Poi(λ = 2)

$$P(X = 5) = e^{-2}\frac{2^5}{5!} \approx 0.0361$$

Geometric Random Variable

- X is **<u>Geometric</u>** Random Variable: X ~ Geo(*p*)
  - X is number of independent trials until first success
  - *p* is probability of success on each trial
  - X takes on values 1, 2, 3, …, with probability:

$$P(X = n) = (1-p)^{n-1}p$$

  - E[X] = 1/*p*   Var(X) = (1 – *p*)/p^2
- Examples:
  - Flipping a fair (*p* = 0.5) coin until first "heads" appears.
  - Urn with N black and M white balls. Draw balls (with replacement, *p* = N/(N + M)) until draw first black ball.
  - Generate bits with P(bit = 1) = *p* until first 1 generated

Negative Binomial Random Variable

- X is **<u>Negative Binomial</u>** RV: X ~ NegBin(*r*, *p*)
  - X is number of independent trials until *r* successes
  - *p* is probability of success on each trial
  - X takes on values *r*, *r* + 1, *r* + 2…, with probability:

$$P(X = n) = \binom{n-1}{r-1}p^r(1-p)^{n-r}, \text{ where } n = r, r+1, ...$$

  - E[X] = *r*/*p*   Var(X) = *r*(1 – *p*)/p^2
- Note: Geo(*p*) ~ NegBin(1, *p*)
- Examples:
  - # of coin flips until *r*-th "heads" appears
  - # of strings to hash into table until bucket 1 has *r* entries

Hypergeometric Random Variable

- X is **<u>Hypergeometric</u>** RV: X ~ HypG(n, N, m)
  - Urn with N balls: (N – m) black and m white.
  - Draw n balls **<u>without</u>** replacement
  - X is number of white balls drawn

$$P(X=i)=\frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}},\ \text{where } i=0,1,\ldots,n$$

  - E[X] = $n(m/N)$ Var(X) = $nm(N-n)(N-m)/N^2(N-1)$
  - Let $p = m/N$ (probability of drawing white on 1st draw)
- Note: HypG(n, N, m) → Bin(n, m/N)
  - As n → ∞ and m/N remains constant

Endangered Species

- Determine N = how many of some species remain
  - Randomly tag m of species (e.g., with white paint)
  - Allow animals to mix randomly (assuming no breeding)
  - Later randomly observe another n of the species
  - X = number of tagged animals in observed group of n
  - X ~ HypG(n, N, m)
- "Maximum Likelihood" estimate
  - Set N to be value that maximizes: $P(X=i)=\frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$

    for the value i of X that you observed → $\hat{N} = mn/i$
- Similar to assuming: i = E[X] = nm/N