From Urns to Coupons

- "Coupon Collecting" is classic probability problem
  - There exist N different types of coupons
  - Each is collected with some probability p_i ($1 \leq i \leq N$)
- Ask questions like:
  - After you collect m coupons, what is probability you have k different kinds?
  - What is probability that you have ≥ 1 of each N coupon types after you collect m coupons?
- You've seen concept (in a more practical way)
  - N coupon types = N buckets in hash table
  - collecting a coupon = hashing a string to a bucket

Digging Deeper on Independence

- Recall, two events E and F are called independent if

  P(EF) = P(E) P(F)

- If E and F are independent, does that tell us anything about:

  P(EF | G) = P(E | G) P(F | G),

  where G is an arbitrary event?

- In general, No!

Not-so Independent Dice

- Roll two 6-sided dice, yielding values D_1 and D_2
  - Let E be event: $D_1 = 1$
  - Let F be event: $D_2 = 6$
  - Let G be event: $D_1 + D_2 = 7$
- E and F are independent
  - P(E) = 1/6, P(F) = 1/6, P(EF) = 1/36
- Now condition both E and F on G:
  - P(E|G) = 1/6, P(F|G) = 1/6, P(EF|G) = 1/6
  - P(EF|G) ≠ P(E|G) P(F|G) → E|G and F|G *dependent*
- Independent events can become dependent by conditioning on additional information

Do CS Majors Get Less A's?

- Say you are in a dorm with 100 students
  - 10 of the students are CS majors: P(CS) = 0.1
  - 30 of the students get straight A's: P(A) = 0.3
  - 3 students are CS majors who get straight A's
    - P(CS, A) = 0.03
    - P(CS, A) = P(CS)P(A), so CS and A are independent
  - At faculty night, only CS majors and A students show up
    - So, 37 (= 10 + 30 – 3) students arrive
    - Of 37 students, 10 are CS ⇒ P(CS | CS or A) = 10/37 = 0.27
    - Appears that being CS major lowers probability of straight A's
    - But, weren't they supposed to be independent?
  - In fact, CS and A <u>conditionally dependent</u> at faculty night

Explaining Away

- Say you have a lawn
  - It gets watered by rain or sprinklers
  - P(rain) and P(sprinklers were on) are independent
  - Now, you come outside and see the grass is wet
    - You know that the sprinklers were on
    - Does that lower probability that rain was cause of wet grass?
  - This phenomena is called "explaining away"
    - One cause of an observation makes other causes less likely
  - Only CS majors and A students come to faculty night
    - Knowing you came because you're a CS major makes it less likely you came because you get straight A's

Conditioning Can Break Dependence

- Consider a randomly chosen day of the week
  - Let A be event: It is not Monday
  - Let B be event: It is Saturday
  - Let C be event: It is the weekend
- A and B are dependent
  - P(A) = 6/7, P(B) = 1/7, P(AB) = 1/7 ≠ (6/7)(1/7)
- Now condition both A and B on C:
  - P(A|C) = 1, P(B|C) = 1/2, P(AB|C) = 1/2
  - P(AB|C) = P(A|C) P(B|C) → A|C and B|C *independent*
- Dependent events can become independent by conditioning on additional information

Conditional Independence

- Two events E and F are called **conditionally independent given G**, if

  P(E F | G) = P(E | G) P(F | G)

  Or, equivalently: P(E | F G) = P(E | G)

- Exploiting conditional independence to generate fast probabilistic computations is one of the main contributions CS has made to probability theory

Random Variable

- A **Random Variable** is a real-valued function defined on a sample space
- Example:
  - 3 fair coins are flipped.
  - Y = number of “heads” on 3 coins
  - Y is a random variable
  - P(Y = 0) = 1/8 (T, T, T)
  - P(Y = 1) = 3/8 (H, T, T), (T, H, T), (T, T, H)
  - P(Y = 2) = 3/8 (H, H, T), (H, T, H), (T, H, H)
  - P(Y = 3) = 1/8 (H, H, H)
  - P(Y ≥ 4) = 0

Binary Random Variables

- A binary random variable is a random variable with 2 possible outcomes
  - *n* coin flips, each which independently come up heads with probability *p*
  - Y = number of “heads” on *n* flips
  - P(Y = k) = $\binom{n}{k} p^k (1-p)^{n-k}$, where *k* = 0, 1, 2, ..., n
  - So, $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$
  - Proof: $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1^n = 1$

Simple Game

- Urn has 11 balls (3 blue, 3 red, 5 black)
  - 3 balls drawn. +$1 for blue, -$1 for red, $0 for black
  - Y = total winnings
  - P(Y = 0) = $\binom{5}{3} + \binom{3}{1}\binom{3}{1}\binom{5}{1} \Big/ \binom{11}{3} = \frac{55}{165}$
  - P(Y = 1) = $\binom{3}{1}\binom{5}{2} + \binom{3}{2}\binom{3}{1} \Big/ \binom{11}{3} = \frac{39}{165}$ = P(Y = -1)
  - P(Y = 2) = $\binom{3}{2}\binom{5}{1} \Big/ \binom{11}{3} = \frac{15}{165}$ = P(Y = -2)
  - P(Y = 3) = $\binom{3}{3} \Big/ \binom{11}{3} = \frac{1}{165}$ = P(Y = -3)

Probability Mass Functions

- A random variable *X* is <u>discrete</u> if it has countably many values (e.g., x_1, x_2, x_3, ...)
- Probability Mass Function (PMF) of a discrete random variable is:

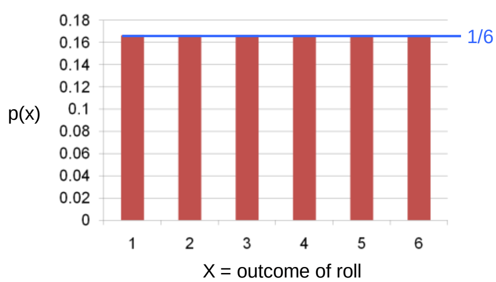
$$p(a) = P(X = a)$$

- Since $\sum_{i=1}^{\infty} p(x_i) = 1$, it follows that:

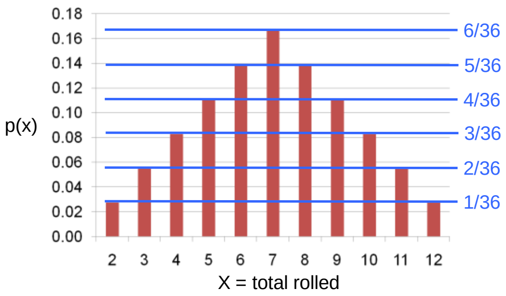
$$P(X = a) = \begin{cases} p(x_i) \geq 0 \text{ for } i = 1, 2, \ldots \\ p(x) = 0 \ \text{ otherwise} \end{cases}$$

  where X can assume values x_1, x_2, x_3, ...

PMF For a Single 6-Sided Die



PMF For a Roll of Two 6-Sided Dice



Cumulative Distribution Functions

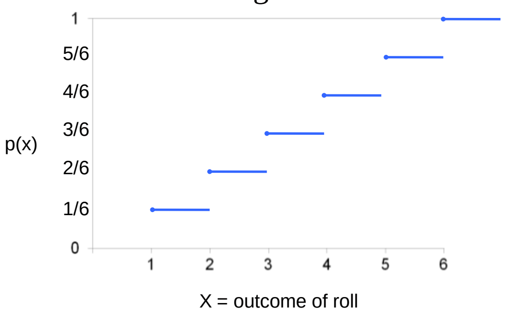
- For a random variable X, the Cumulative Distribution Function (CDF) is defined as:

$$F(a) = F(X \le a) \quad \text{where } -\infty < a < \infty$$

- The CDF of a discrete random variable is:

$$F(a) = F(X \le a) = \sum_{\text{all } x \le a} p(x)$$

CDF For a Single 6-Sided Die



Expected Value

- The Expected Values for a discrete random variable X is defined as:

$$E[X] = \sum_{x:p(x)>0} x\, p(x)$$

- Note: sum over all values of x that have p(x) > 0.
- Expected value also called: *Mean*, *Expectation*, *Weighted Average*, *Center of Mass*, *1st Moment*

Expected Value Examples

- Roll a 6-Sided Die. X is outcome of roll
  - p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = 1/6
- E[X] = $1\left(\frac{1}{6}\right)+2\left(\frac{1}{6}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)+5\left(\frac{1}{6}\right)+6\left(\frac{1}{6}\right)=\frac{7}{2}$
- Y is random variable
  - P(Y = 1) = 1/3, P(Y = 2) = 1/6, P(Y = 3) = 1/2
- E[Y] = 1 (1/3) + 2 (1/6) + 3 (1/2) = 13/6

Indicator Variables

- A variable I is called an indicator variable for event *A* if

$$I = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A^c \text{ occurs} \end{cases}$$

- What is E[I]?
  - p(1) = P(A), p(0) = 1 – P(A)
  - E[I] = 1 P(A) + 0 (1 – P(A)) = P(A)

Lying With Statistics

"There are three kinds of lies:
lies, damned lies, and statistics"

– Mark Twain

- School has 3 classes with 5, 10 and 150 students
- Randomly choose a class with equal probability
- X = size of chosen class
- What is E[X]?
  - E[X] = 5 (1/3) + 10 (1/3) + 150 (1/3)
    = 165/3 = 55

Lying With Statistics

"There are three kinds of lies:
lies, damned lies, and statistics"

– Mark Twain

- School has 3 classes with 5, 10 and 150 students
- Randomly choose a student with equal probability
- Y = size of class that student is in
- What is E[Y]?
  - E[Y] = 5 (5/165) + 10 (10/165) + 150 (150/165)
    = 22635/165 ≈ 137
- Note: E[Y] is students' perception of class size
  - But E[X] is what is usually reported by schools!

Expectation of a Random Variable

- Let Y = g(X), where g is real-valued function

$$E[g(X)] = E[Y] = \sum_j y_j p(y_j) = \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i)$$
$$= \sum_j g(x_i) \sum_{i:g(x_i)=y_j} p(x_i) = \sum_j \sum_{i:g(x_i)=y_j} g(x_i) p(x_i)$$
$$= \sum_i g(x_i) p(x_i)$$

Other Properties of Expectations

- Linearity:

$$E[aX+b] = aE[X]+b$$

  - Consider X = 6-sided die roll, Y = 2X – 1.
  - E[X] = 3.5 E[Y] = 6
- *N*-th Moment of X:

$$E[X^n] = \sum_{x:\,p(x)>0} x^n p(x)$$

  - We'll see the 2^{nd} moment soon...

Utility

- Utility is value of some choice
  - 2 choices, each with n consequences: $c_1, c_2, \ldots, c_n$
  - One of c_i will occur with probability p_i
  - Each consequence has some value (utility): $U(c_i)$
  - Which choice do you make?
- Example: Buy a \$1 lottery ticket (for \$1M prize)?
  - Probability of winning is $1/10^7$
  - **Buy**: c_1 = win, c_2 = lose, $U(c_1) = 10^6 - 1$, $U(c_2) = -1$
  - **Don't Buy**: c_1 = lose, $U(c_1) = 0$
  - E(buy) = $1/10^7 (10^6 - 1) + (1 - 1/10^7)(-1) \approx -0.9$
  - E(don't buy) = 1 (0) = 0
  - *"You can't lose if you don't play!"*