CSE 312: Foundations of Computing II

Section 1: Combinatorics

1. Summary of Main Concepts (Counting)

(a) **Sum Rule:** If an experiment can either end up having one of N outcomes, or one of M outcomes (where there is no overlap), then the total number of possible outcomes is $N + M$.

(b) **Product Rule:** If an object or an outcome can be selected by a sequence of choices, where there are m_1 possibilities for the first choice, m_2 possibilities for the second choice (given the first), m_3 possibilities for the third choice (given the first two) and so on up to the k-th choice, then there are $m_1 \cdot m_2 \cdot m_3 \cdots m_k = \prod_{i=1}^{k} m_i$ possible outcomes overall.

(c) **Number of ways to order n distinct objects:** $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

(d) **Number of ways to select from n distinct objects:**

(a) **Permutations** (number of ways to linearly arrange k objects out of n distinct objects, when the order of the k objects matters):

$$P(n, k) = \frac{n!}{(n-k)!}$$

(b) **Combinations** (number of ways to choose k objects out of n distinct objects, when the order of the k objects does not matter):

$$\frac{n!}{k!(n-k)!} = \binom{n}{k} = C(n, k)$$

(e) **Complementary Counting (Complementing):** If asked to find the number of ways to do X, you can: find the total number of ways and then subtract the number of ways to not do X.

The rest of these should be covered tomorrow and Monday.

(f) **Binomial Theorem:** $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

(g) **Principle of Inclusion-Exclusion (PIE)** : For 2 events, it says $|A \cup B| = |A| + |B| - |A \cap B|$
For 3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
In general: +singles - doubles + triples - quads + ...

(h) **Multinomial coefficients**: Suppose there are n objects, but only k are distinct, with $k \leq n$. (For example, "godoggy" has $n = 7$ objects (characters) but only $k = 4$ are distinct: (g, o, d, y)). Let n_i be the number of times object i appears, for $i \in \{1, 2, \ldots, k\}$. (For example, $(3, 2, 1, 1)$, continuing the "godoggy" example.) The number of distinct ways to arrange the n objects is:

$$\frac{n!}{n_1! n_2! \cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}$$

2. Seating

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if . . .

(a) . . . all couples are to get adjacent seats?

(b) . . . anyone can sit anywhere, except that one particular couple just had a fight and insists on *not* sitting in adjacent seats?

3. Weird Card Game

In how many ways can a pack of fifty-two cards be dealt to thirteen players, four to each, so that every player has one card of each suit?

4. Escape the Professor

There are 6 security professors and 7 theory professors taking part in an escape room. If 4 security professors and 4 theory professors are chosen and paired off, how many pairings are possible?

For this section, we expect to end here (or before!). The rest of these problems can be done at home for extra practice, or if you finish these early. Solutions will be posted.

5. HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

6. Birthday Cake

A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert menus are there for the week?

7. Full Class

There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

8. Paired Finals

Suppose you are to take a CSE 312 final in pairs. There are 100 students in the class and 8 TAs, so 8 lucky students will get to pair up with a TA. Each TA must take the exam with some student, but two TAs cannot take the exam together. How many ways can they pair up?

9. Photographs

Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

10. Rabbits!

Rabbits Peter and Pauline have three offspring: Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

11. Extended Family Portrait

A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the nm people be arranged if members of a family must stay together?

12. Subsubset

Let $[n] = \{1, 2, ..., n\}$ denote the first n natural numbers. How many (ordered) pairs of subsets (A, B) are there such that $A \subseteq B \subseteq [n]$?

13. Divide Me

How many numbers in $[360]$ are divisible by:

(a) 4, 6, and 9?

(b) 4, 6, or 9?

(c) Neither 4, 6, nor 9?

14. Binomials

What is the coefficient of z^{36} in $(-2x^2yz^3 + 5uv)^{312}$?