CSE 312: Foundations of Computing II

Section 2: Intro Probability Solutions

1. Review of Main Concepts

(a) **Binomial Theorem:** $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

(b) **Principle of Inclusion-Exclusion (PIE)**: For 2 events, it says $|A \cup B| = |A| + |B| - |A \cap B|$
For 3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
In general: +singles - doubles + triples - quads + ...

(c) **Complementary Counting (Complementing):** If asked to find the number of ways to do X, you can: find the total number of ways and then subtract the number of ways to not do X.

(d) **Multinomial coefficients**: Suppose there are n objects, but only k are distinct, with $k \leq n$. (For example, "godoggy" has $n = 7$ objects (characters) but only $k = 4$ are distinct: (g, o, d, y)). Let n_i be the number of times object i appears, for $i \in \{1, 2, \ldots, k\}$. (For example, $(3, 2, 1, 1)$, continuing the "godoggy" example.) The number of distinct ways to arrange the n objects is:

$$\frac{n!}{n_1! n_2! \cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}$$

(e) **Pigeonhole Principle**: Suppose there are $n-1$ pigeon holes and n pigeons, and each pigeon goes into a hole. Then, there must be some hole that has two pigeons in it. This simple observation is surprisingly useful in computer science.

We can put this more generally as: if there are n pigeons and k holes, and $n > k$, some hole has at least $\lceil \frac{n}{k} \rceil$ pigeons.

For the pigeon haters out there, we can also express this as "we have n holes and $n-1$ pigeons...". Pick your favorite.

(f) **Combinatorial proof:** Prove identity by showing that there are two different ways of counting some set of objects.

(g) **Key Probability Definitions**

(a) **Sample Space:** The set of all possible outcomes of an experiment, denoted Ω or S
(b) **Event:** Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
(c) **Union:** The union of two events E and F is denoted $E \cup F$
(d) **Intersection:** The intersection of two events E and F is denoted $E \cap F$ or EF
(e) **Mutually Exclusive:** Events E and F are mutually exclusive iff $E \cap F = \emptyset$
(f) **Complement:** The complement of an event E is denoted E^C or $\overline{E}$ or $\neg E$, and is equal to $\Omega \setminus E$
(g) **DeMorgan's Laws:** $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$
(h) **Probability of an event E:** denoted $\mathbb{P}(E)$ or $\Pr(E)$ or $P(E)$
(i) **Partition:** Nonempty events $E_1, \ldots, E_n$ partition the sample space Ω iff
- $E_1, \ldots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$, and
- $E_1, \ldots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$
  - Note that for any event A (with $A \neq \emptyset, A \neq \Omega$): A and A^C partition Ω

(h) **Axioms of Probability and their Consequences**

  (a) **Axiom 1: Non-negativity** For any event E, $\mathbb{P}(E) \geq 0$

  (b) **Axiom 2: Normalization** $\mathbb{P}(\Omega) = 1$

  (c) **Axiom 3: Countable Additivity** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$. Also, if $E_1, E_2, ...$ is a countable sequence of disjoint events, $\mathbb{P}(\bigcup_{k=1}^{\infty} E_i) = \sum_{k=1}^{\infty} \mathbb{P}(E_i)$.

  (d) **Corollary 1: Complementation** $\mathbb{P}(E) + \mathbb{P}(E^C) = 1$

  (e) **Corollary 2: Monotonicity** If $E \subseteq F$, $\mathbb{P}(E) \leq \mathbb{P}(F)$

  (f) **Corollary 2: Inclusion-Exclusion** $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

(i) **Equally Likely Outcomes:** If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) = \dfrac{|E|}{|\Omega|}$.

  - Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.

2. Probability by Simulation

In section, we'll work through this Edstem lesson on Probability by Simulation. This will be very helpful for the coding portion of Pset 2.

For problems 3-5 and 8, first answer the following two questions and then answer the question stated. (i) What is the sample space and how big is it? (ii) What is the probability of each outcome in the sample space? Unless otherwise specified, each outcome is equally likely.

3. Spades and Hearts

Given 3 different spades and 3 different hearts, shuffle them. Compute Pr(E), where E is the event that the suits of the shuffled cards are in alternating order.

Solution:

The sample space Ω is all re-orderings possible: there are $|\Omega| = 6!$ such. Now for E, order the spades and hearts independently, so there are $3!^2$ ways to do so. Finally choose whether you want hearts or spades first. All such orderings are equally likely, so Pr$(E) = \frac{|E|}{|\Omega|} = \frac{2 \cdot 3!^2}{6!}$.

4. Trick or Treat

Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly N total candies. You count that there are exactly K of them which are kit kats (and the rest are not). The sign says that each kid should take exactly n candies. Suppose that when the next kid shows up, they draw n candies, with each subset of size n equally likely to be drawn. Let X be the number of kit kats the kid draws (out of n). What is Pr$(X = k)$, that is, the probability the kid draws exactly k kit kats?

Solution:

Let E be the event that $X = k$, and the sample space be every way we can choose n candies out of N total.

$$\boxed{\Pr(X = k) = \frac{|E|}{|\Omega|} = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}}$$

In order to choose exactly k kit kats, we must choose k out of the K kit kats, and $n-k$ out of the $N-K$ other candies. The size of the sample space is just the number of ways to choose n candies out of N.

5. Staff Photo

Suppose we have 11 chairs (in a row) with 7 TA's, and Professors Karlin, Lin, Tessaro, and Rao to be seated. Suppose all seatings are equally likely. What is the probability that every professor has a TA to their immediate left and right?

Solution:

Imagine we permute all 7 TA's first – there are $7!$ ways to do this. Then, there are 6 spots between them that would result in a TA on both sides. We choose 4 of them for the Professors to sit – order matters since each Professor is distinct, so we multiply by $4!$. So the total ways is $7! \cdot \binom{6}{4} \cdot 4!$.

The sample space is the total number of ways to seat all 11 people: simply $11!$.

Since each seating is equally likely, the probability is then $\boxed{\dfrac{7! \cdot \binom{6}{4} \cdot 4!}{11!}}$

6. A Team and a Captain

Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r}r.$$

Hint: Consider two ways to choose a team of size r out of a set of size n and a captain of the team (who is also one of the team members).

Solution:

Remember that a combinatorial proof just requires that we show both sides are equivalent ways of counting a situation.

Left hand side: Choose a team of size r and a captain for that team (from among the r) by first choosing the captain (n choices) and then choosing the rest of the team $\binom{n-1}{r-1}$.

Right hand side: Choose a team of size r and a captain for that team by first choosing the team ($\binom{n}{r}$ choices) and then choosing the captain from among the members of the team (r choices).

7. Weighted Die

Consider a weighted die such that

- $\Pr(1) = \Pr(2)$,
- $\Pr(3) = \Pr(4) = \Pr(5) = \Pr(6)$, and
- $\Pr(1) = 3\Pr(3)$.

What is the probability that the outcome is 3 or 4?

Solution:

By the second axiom of probability, the sum of probabilities for the sample space must equal 1. That is, $\sum_{i=1}^{6} \Pr(i) = 1$. Since $\Pr(1) = \Pr(2)$ and $\Pr(1) = 3\Pr(3)$, we have that: $1 = \Pr(1) + \Pr(2) + \Pr(3) + \Pr(4) + \Pr(5) + \Pr(6) = 3\Pr(3) + 3\Pr(3) + \Pr(3) + \Pr(3) + \Pr(3) + \Pr(3) = 10\Pr(3)$

Thus, solving algebraically, $\Pr(3) = 0.1$, so $\Pr(3) = \Pr(4) = 0.1$. Since rolling a 3 and 4 are disjoint events, then $\Pr(3 \text{ or } 4) = \Pr(3) + \Pr(4) = 0.1 + 0.1 = 0.2$.

8. Fleas on Squares (Pigeonhole principle)

25 fleas sit on a 5×5 checkerboard, one per square. At the stroke of noon, all jump across an edge (not a corner) of their square to an adjacent square. At least two must end up in the same square. Why?

Solution:

There are two colors on a checkerboard; 13 are of one color, and 12 are of another. Each colored square is only surrounded by opposite colored squares on its edges. Therefore, the 13 fleas on the first color can only jump to a square of the second color — of which there are only 12 positions. So at least two fleas must land on the same square by the pigeonhole principle.

9. PigONEholes

Let $k \geq 2$ be some integer. Show that there exists a positive integer n consisting of only digits $0, 1$ and no larger than 10^{k+2} such that $k|n$. (Hint: Consider the sequence of length $k+1$ of $1, 11, 111, 1111, ...$).

Solution:

Consider the sequence of numbers of length $k+1$: $1, 11, 111, ...,$, such that the j^{th} element is the number consisting of exactly j 1's. Take these numbers mod k. Since there are $k+1$ numbers and k possible remainders, two have the same remainder. Call the larger one b and the smaller a. Their difference $n = b - a$ must be divisible by k, and consist of only 1's and 0's.

10. Ingredients

(a) Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent.

Solution:

We use inclusion-exclusion. Let Ω be the set of all anagrams (permutations) of "INGREDIENT", and A_I be the set of all anagrams with two consecutive I's. Define A_E and A_N similarly. $A_I \cup A_E \cup A_N$ clearly are the set of anagrams we don't want. So we use complementing to count the size of $\Omega \setminus (A_I \cup A_E \cup A_N)$. By inclusion exclusion, $|A_I \cup A_E \cup A_N|$ =singles-doubles+triples, and by complementing, $|\Omega \setminus (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|$.

First, $|\Omega| = \frac{10!}{2!2!2!}$ because there are 2 of each of I,E,N's (multinomial coefficient). Clearly, the size of A_I is the same as A_E and A_N. So $|A_I| = \frac{9!}{2!2!}$ because we treat the two adjacent I's as one entity. We also need $|A_I \cap A_E| = \frac{8!}{2!}$ because we treat the two adjacent I's as one entity and the two adjacent E's as one entity (same for all doubles). Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I's, E's, and N's as one entity.

Putting this together gives $\boxed{\frac{10!}{2!2!2!} - \left(\binom{3}{1} \cdot \frac{9!}{2!2!} - \binom{3}{2} \cdot \frac{8!}{2!} + \binom{3}{3} \cdot 7!\right)}$

(b) Repeat the question for the letters "AAAAABBB".

Solution:

For the second question, note that no A's and no B's can be adjacent. So let us put the B's down first: $\_B\_B\_B\_$

By the pigeonhole principle, two A's must go in the same slot, but then they would be adjacent, so there are $\boxed{\text{no ways}}$.

11. Acing the Exams

In a town of 351 students (the number of students, not ones taking CSE 351), every student aces the midterm, final, or both. If 331 of the students ace the midterm and 45 ace the final, what is the number of students who aced the midterm but did not ace the final as well?

Solution:

By inclusion-exclusion, the number of people who aced both the midterm and the final is $331 + 45 - 351 = 25$. For one of the 331 students who aced the midterm; either they aced the final or they didn't, so $331 - 25 = 306$ did not ace the final.

12. Divisibility

Consider the set $T = \{1, 2, ..., 36050\}$, and suppose we choose a subset S of size 3605, each equally likely. What is the probability that there are two (distinct) numbers in S whose difference is divisible by 99?

Solution:

This probability is 1 by the pigeonhole principle. Consider each of the elements of S mod 99 (there are 99 possible remainders $0, ..., 98$). By the pigeonhole principle, since $3605 > 99$, there are at least two with the same remainder. Take those two numbers, and their difference is divisible by 99.

13. Senate Committee Assignments

There are 51 senators in a senate. The senate needs to be divided into n committees such that each senator is on exactly one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does 'not' necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

Solution:

We'll prove this statement by induction: for any number of senators who hate exactly three people, 7 committees is enough. (and hence, in particular for 51 senators.)

Base Case: For $n \leq 7$, this is trivial - assign everyone to their own committee.

Induction Hypothesis: Now suppose it is true for any group of n senators ; we will show it is true for $n + 1$ senators.

Induction Step: We claim that there exists at least one senator who is hated by at most three people. For if not, all senators are hated by > 3 people, a contradiction since there are exactly $3n$ pairs (A, B), where A hates B. (This is using the Pigeonhole Principle).

Choose one such senator S and separate that senator. For the other n senators, the induction hypothesis says we can arrange them in 7 committees. But senator S hates exactly three people, and is hated by at most three people, so can only be excluded from at most six committees. Place S in one of the remaining committees, and this concludes our induction step.

Minimality: Suppose $n \geq 7$. To show 7 is the least such number, if we only had 6 committees, imagine three senators hating some senator S, and S hating a different set of three senators. Then, S cannot be placed in any committee.

14. Congressional Tea Party

Twenty politicians are having a tea party, 6 Democrats and 14 Republicans.

(a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

Solution:

The sample space is the number of ways to give tea to people, so there are $\binom{20}{10}$ ways. The event is the ways to give tea to only Republicans, of which there are $\binom{14}{10}$ ways. So the probability is $\frac{\binom{14}{10}}{\binom{20}{10}}$.

(b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 8 Republicans and 2 Democrats?

Solution:

Similarly to the previous part, $\frac{\binom{14}{8}\binom{6}{2}}{\binom{20}{10}}$.

15. Friendly Proofs

Show that in a group of n people (who may be friends with any number of other people), two must have the same number of friends.

Solution:

Say there are k people with 0 friends in the group. If $k \geq 2$, we are done. Otherwise, the remaining $n-k$ people can have between 1 and $n-k-1$ friends. Now apply the Pigeonhole Principle: map each person to the number of friends they have. Since $n-k$ pigeons are mapped to $n-k-1$ pigeonholes, there must be two with the same number of friends.

16. Count the Solutions

Consider the following equation: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 70$. A solution to this equation over the nonnegative integers is a choice of a nonnegative integer for each of the variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the equation. For example, $a_1 = 15, a_2 = 3, a_3 = 15, a_4 = 0, a_5 = 7, a_6 = 30$ is a solution. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the equation?

Solution:

(Hint: Think about splitting a sequence of 70 1's into 6 blocks, each block consisting of consecutive 1's in the sequence. The number of 1's in the i-th block corresponds to the value of a_i. Note that the i-th block is allowed to be empty, corresponding to $a_i = 0$.)
Using the stars and bars method, we get:

$$\binom{70+6-1}{6-1} = \binom{75}{5} = 17,259,390$$