CSE 312: Foundations of Computing II

Section 7: Joint Distributions, Law of Total Expectation (and bit of conditional distributions) Solutions

1. Review of Main Concepts

(a) **Multivariate: Discrete to Continuous:**

	Discrete	**Continuous**
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X=x, Y=y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x, s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Independence must have	$\forall x,y,\ p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x,y,\ f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

(b) **Law of Total Probability (r.v. version):** If X is a discrete random variable, then

$$\mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A | X = x) p_X(x) \qquad \text{discrete } X$$

(c) **Law of Total Expectation (Event Version)**: Let X be a discrete random variable, and let events $A_1, ..., A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \, \mathbb{P}(A_i)$$

(d) **Conditional Expectation**: See table bbelow. Note that linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y \mid A] = \mathbb{E}[X \mid A] + \mathbb{E}[Y \mid A]$

(e) **Law of Total Expectation (RV Version)**: Suppose X and Y are random variables. Then,

$$\mathbb{E}[X] = \sum_{y} \mathbb{E}[X \mid Y = y] \, p_Y(y) \qquad \text{discrete version.}$$

(f) **Conditional distributions (not realy covered in class)**

	Discrete	**Continuous**
Conditional PMF/PDF	$p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$\mathbb{E}[X \mid Y = y] = \sum_x x p_{X\|Y}(x\|y)$	$\mathbb{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y) dx$

(g) **The following have not been covered as of 11/17:**

- Law of Total Probability (continuous)

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A | X = x) f_X(x) dx$$

- Law of total expectation (continuous)

$$\mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X \mid Y = y] \, f_Y(y) dy$$

2. Joint PMF's

Suppose X and Y have the following joint PMF:

X/Y	1	2	3
0	0	0.2	0.1
1	0.3	0	0.4

(a) Identify the range of X (Ω_X), the range of Y (Ω_Y), and their joint range ($\Omega_{X,Y}$).

Solution:

$\Omega_X = \{0, 1\}$, $\Omega_Y = \{1, 2, 3\}$, and $\Omega_{X,Y} = \{(0, 2), (0, 3), (1, 1), (1, 3)\}$

(b) Find the marginal PMF for X, $p_X(x)$ for $x \in \Omega_X$.

Solution:

$$p_X(0) = \sum_y p_{X,Y}(0, y) = 0 + 0.2 + 0.1 = 0.3$$

$$p_X(1) = 1 - p_X(0) = 0.7$$

(c) Find the marginal PMF for Y, $p_Y(y)$ for $y \in \Omega_Y$.

Solution:

$$p_Y(1) = \sum_x p_{X,Y}(x, 1) = 0 + 0.3 = 0.3$$

$$p_Y(2) = \sum_x p_{X,Y}(x, 2) = 0.2 + 0 = 0.2$$

$$p_Y(3) = \sum_x p_{X,Y}(x, 3) = 0.1 + 0.4 = 0.5$$

(d) Are X and Y independent? Why or why not?

Solution:

No, since a necessary condition is that $\Omega_{X,Y} = \Omega_X \times \Omega_Y$.

(e) Find $\mathbb{E}\left[X^3 Y\right]$.

Solution:

Note that $X^3 = X$ since X takes values in $\{0, 1\}$.

$$\mathbb{E}\left[X^3 Y\right] = \mathbb{E}[XY] = \sum_{(x,y)\in\Omega_{X,Y}} xy p_{X,Y}(x, y) = 1 \cdot 1 \cdot 0.3 + 1 \cdot 3 \cdot 0.4 = 1.5$$

3. Trinomial Distribution

A generalization of the Binomial model is when there is a sequence of n independent trials, but with three outcomes, where $\mathbb{P}(\text{outcome } i) = p_i$ for $i = 1, 2, 3$ and of course $p_1 + p_2 + p_3 = 1$. Let X_i be the number of times outcome i occurred for $i = 1, 2, 3$, where $X_1 + X_2 + X_3 = n$. Find the joint PMF $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ and specify its value for all $x_1, x_2, x_3 \in \mathbb{R}$.

Solution:

Same argument as for the binomial PMF:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \binom{n}{x_1, x_2, x_3} \prod_{i=1}^{3} p_i^{x_i} = \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

where $x_1 + x_2 + x_3 = n$ and are nonnegative integers.

4. Do You "Urn" to Learn More About Probability?

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the i-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

(a) X_1, X_2

(b) X_1, X_2, X_3

5. Successes

Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first 2 successes. Find the joint pmf of X_1 and X_2. Write an expression for $E[\sqrt{X_1 X_2}]$. You can leave your answer in the form of a sum.

Solution:

X_1 and X_2 take on two particular values x_1 and x_2, when there are x_1 failures followed by one success, and then x_2 failures followed by one success. Since the Bernoulli trials are independent the joint pmf is

$$p_{X_1,X_2}(x_1, x_2) = (1-p)^{x_1} p \cdot (1-p)^{x_2} p = (1-p)^{x_1+x_2} p^2$$

for $(x_1, x_2) \in \Omega_{X_1,X_2} = \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$. By the definition of expectation

$$E[\sqrt{X_1 X_2}] = \sum_{(x_1,x_2) \in \Omega_{X_1,X_2}} \sqrt{x_1 x_2} \cdot (1-p)^{x_1+x_2} p^2.$$

6. Continuous joint density I

The joint probability density function of X and Y is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) & 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that this is indeed a joint density function.

(b) Compute the marginal density function of X.

(c) Find $Pr(X > Y)$. (Uses the continuous law of total probability which we have not covered in class as of 11/17.)

(d) Find $P(Y > \frac{1}{2} | X < \frac{1}{2})$.

(e) Find $E(X)$.

(f) Find $E(Y)$

Solution:

(a) A joint density function will integrate to 1 over all possible values. Thus, we integrate over the joint range range using Wolfram Alpha, and see that:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = \int_0^2\int_0^1 \frac{6}{7}(x^2+\frac{xy}{2})dxdy = 1$$

We also need to check that the density is nonnegative, but that is easily seen to be true.

(b) We apply the definition of the marginal density function of X, using the fact that we only need to integrate over the values where the joint density is positive:

$$f_X(x) = \begin{cases} \int_0^2 \frac{6}{7}(x^2+\frac{xy}{2})dy = \frac{6}{7}x(2x+1) & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) First, we rearrange our initial probability. Then, by the continuous law of total probability:

$$\mathbb{P}(X > Y) = 1 - \mathbb{P}(X \le Y) = 1 - \int_{-\infty}^{\infty} \mathbb{P}(X \le Y | Y = y) f_Y(y)dy = 1 - \int_{-\infty}^{\infty} \mathbb{P}(X \le y) f_Y(y)dy$$

Once again, we can instead integrate over just the range of y, getting:

$$1 - \int_0^2 \mathbb{P}(X \le y) f_Y(y)dy$$

We have to remember that $f_X(x)$ is positive only when $0 < x < 1$. Thus, $F_X(x) = 1$ for $x \ge 1$, so we have:

$$1 - \int_0^1 \mathbb{P}(X \le y) f_Y(y)dy - \int_1^2 f_Y(y)dy$$

So, now we just need to find the CDF of X, and the marginal PDF of Y. For the former, for any $0 < x < 1$, we have

$$F_X(x) = \int_0^x \frac{6}{7}u(2u+1)du = \frac{1}{7}x^2(4x+3)$$

For the latter, for $0 < y < 2$, we have

$$f_Y(y) = \int_0^1 \frac{6}{7}(x^2+\frac{xy}{2})dx = \frac{1}{14}(3y+4)$$

Putting these together, we get that:

$$\mathbb{P}(X > Y) = 1 - \int_0^1 \frac{1}{7}y^2(4y+3)\frac{1}{14}(3y+4)dy - \int_1^2 \frac{1}{14}(3y+4)dy = 1 - \frac{253}{1960} - \frac{17}{28} = \frac{517}{1960} = 0.2638$$

(d) By the definition of conditional probability:

$$\mathbb{P}\left(Y > \frac{1}{2} | X < \frac{1}{2}\right) = \frac{\mathbb{P}(Y > \frac{1}{2},\ X < \frac{1}{2})}{\mathbb{P}(X < \frac{1}{2})}$$

For the numerator, we have

$$\mathbb{P}(Y > \frac{1}{2}, X < \frac{1}{2}) = \int_{1/2}^{\infty}\int_{-\infty}^{1/2} f_{X,Y}(x,y)dxdy$$

$$= \int_{1/2}^2\int_0^{1/2} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)dxdy = \frac{69}{448}$$

For the denominator, we can integrate using the marginal distribution that we found before:

$$\int_0^{1/2} \frac{6}{7}x(2x+1)dx = \frac{5}{28}$$

Putting these together, we get:

$$\mathbb{P}(Y > \frac{1}{2}|X < \frac{1}{2}) = \frac{\frac{69}{448}}{\frac{5}{28}} = 0.8625$$

(e) By definition, and using $\Omega_X = (0,1)$:

$$\mathbb{E}[X] = \int_0^1 f_X(x)xdx = \int_0^1 \frac{6}{7}x(2x+1)xdx = \frac{5}{7}$$

(f) By definition, and using $\Omega_Y = (0,2)$:

$$\mathbb{E}[Y] = \int_0^2 f_Y(y)ydy = \int_0^2 \frac{1}{14}(3y+4)ydy = \frac{8}{7}$$

7. Continuous joint density II

The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

and the joint density of W and V is given by

$$f_{W,V}(w,v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent? Are W and V independent?

Solution:

For two random variables X, Y to be independent, we must have $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x \in \Omega_X$, $y \in \Omega_X$. Let's start with X and Y by finding their marginal PDFs. By definition, and using the fact that the joint PDF is 0 outside of $y > 0$, we get:

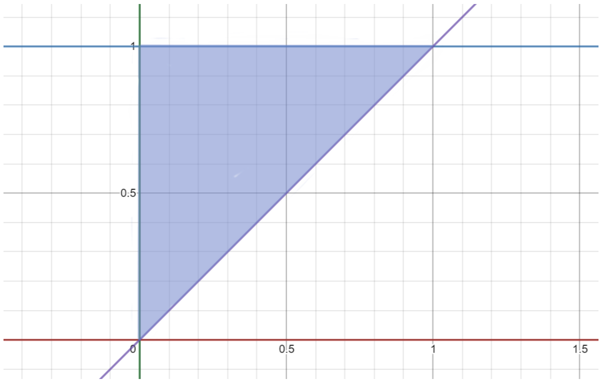
$$f_X(x) = \int_0^\infty xe^{-(x+y)}dy = e^{-x}x$$

We do the same to get the PDF of Y, again over the range $x > 0$:

$$f_Y(y) = \int_0^\infty xe^{-(x+y)}dx = e^{-y}$$

Since $e^{-x}x \cdot e^{-y} = xe^{-x-y} = xe^{-(x+y)}$ for all $x, y > 0$, X and Y are independent.

We can see that W and V are not independent simply by observing that $\Omega_W = (0,1)$ and $\Omega_V = (0,1)$, but $\Omega_{W,V}$ is not equal to their Cartesian product. Specifically, looking at their range of $f_{W,V}(w,v)$. Graphing it with w as the "x-axis" and v as the "y-axis", we see that :



The shaded area is where the joint pdf is strictly positive. Looking at it, we can see that it is not rectangular, and therefore it is not the case that $\Omega_{W,V} = \Omega_W \times \Omega_V$. Remember, the joint range being the Cartesian product of the marginal ranges is not sufficient for independence, but it is *necessary*. Therefore, this is enough to show that they are not independent.

8. Trapped Miner

A miner is trapped in a mine containing 3 doors.

- D_1: The 1^{st} door leads to a tunnel that will take him to safety after 3 hours.
- D_2: The 2^{nd} door leads to a tunnel that returns him to the mine after 5 hours.
- D_3: The 3^{rd} door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters $(12, \frac{1}{3})$.

At all times, he is equally likely to choose any one of the doors. What is the expected number of hours for this miner to reach safety?

Solution:

Let T = number of hours for the miner to reach safety. (T is a random variable)
Let D_i be the event the i^{th} door is chosen. $i \in \{1, 2, 3\}$. Finally, let T_3 be the time it takes to return to the mine in the third case only (a random variable). Note that the expectation of T_3 is $12 * \frac{1}{3}$ because it is binomially distributed with parameters $n = 12, p = \frac{1}{3}$. By Law of Total Expectation, linearity of expectation, and by applying the conditional expectations given by the problem statement:

$$\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[T \mid D_1]\,\mathbb{P}(D_1) + \mathbb{E}[T \mid D_2]\,\mathbb{P}(D_2) + \mathbb{E}[T \mid D_3]\,\mathbb{P}(D_3) \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3 + T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3] + \mathbb{E}[T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (4 + \mathbb{E}[T]) \cdot \frac{1}{3}
\end{aligned}$$

Solving this equation for $\mathbb{E}[T]$, we get

$$\mathbb{E}[T] = 12$$

Therefore, the expected number of hours for this miner to reach safety is 12.

9. Elevator

[We have done this problem in class.] The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers. Assume an infinitely large elevator.

Solution:

Let X = number of people who enter the elevator. $X \sim Poi(10)$. Let Y = number of stops. (X, Y are both random variables)

$$\mathbb{E}[Y] = \sum_{k=0}^{\infty} E[Y|X=k]\mathbb{P}(X=k)$$

$$\mathbb{P}(X=k) = e^{-\lambda}\frac{\lambda^k}{k!} = e^{-10}\frac{10^k}{k!}$$

Let Y_i be an indicator random variable. $Y_i =$ whether the elevator stops at the i^{th} floor.

$$\mathbb{E}[Y \mid X=k] = E[Y_1 + Y_2 + ... + Y_N|X=k]$$

$$\mathbb{E}[Y_i \mid X=k] = 1 - (\frac{N-1}{N})^k$$

By linearity of expectation:

$$\mathbb{E}[Y \mid X=k] = \sum_{i=1}^{N} \mathbb{E}[Y_i \mid X=k] = N \cdot (1 - (\frac{N-1}{N})^k)$$

Finally, we put everything together:

$$\mathbb{E}[Y] = \sum_{k=0}^{\infty} (N \cdot (1 - (\frac{N-1}{N})^k)) \cdot (e^{-10}\frac{10^k}{k!})$$

10. Lemonade Stand

Suppose I run a lemonade stand, which costs me \$100 a day to operate. I sell a drink of lemonade for \$20. Every person who walks by my stand either buys a drink or doesn't (no one buys more than one). If it is raining, n_1 people walk by my stand, and each buys a drink independently with probability p_1. If it isn't raining, n_2 people walk by my stand, and each buys a drink independently with probability p_2. It rains each day with probability p_3, independently of every other day. Let X be my profit over the next week. In terms of n_1, n_2, p_1, p_2 and p_3, what is $\mathbb{E}[X]$?

Solution:

Let R be the event it rains. Let X_i be how many drinks I sell on day i for $i = 1, ..., 7$. We are interested in $X = \sum_{i=1}^{7}(20X_i - 100)$. We have $X_i|R \sim$ Binomial(n_1, p_1), so $\mathbb{E}[X_i \mid R] = n_1p_1$. Similarly, $X_i|R^C \sim$ Binomial(n_2, p_2), so $\mathbb{E}\big[X_i \mid R^C\big] = n_2p_2$. By the law of total expectation,

$$\mu = \mathbb{E}[X_i] = \mathbb{E}[X_i \mid R]\,\mathbb{P}(R) + \mathbb{E}\big[X_i \mid R^C\big]\,\mathbb{P}(R^C) = n_1p_1p_3 + n_2p_2(1-p_3)$$

Hence, by linearity of expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{7}(20X_i - 100)\right] = 20\sum_{i=1}^{7}\mathbb{E}[X_i] - 700 = 140\mu - 700$$

$$= 140 \cdot (n_1 p_1 p_3 + n_2 p_2 (1 - p_3)) - 700.$$

11. Particle Emissions

Suppose we are measuring particle emissions, and the number of particles emitted follows a Poisson distribution with parameter λ, $X \sim \text{Poisson}(\lambda)$. Suppose our device to measure emissions is not always entirely accurate sometimes we fail to observe particles that actually emitted. So for each particle actually emitted, say we have probability p of actually recording it, independently of other particles. Let Y be the number of particles we observed. What distribution does Y follow with what parameters, and what is $\mathbb{E}[Y]$?

Solution:

(We more or less did this problem in class.)

$$\begin{aligned}
p_Y(y) &= \mathbb{P}(Y = y) \\
&= \sum_{x=y}^{\infty} \mathbb{P}(Y = y | X = x) \mathbb{P}(X = x) \quad \text{(Law of Total Probability)} \\
&= \sum_{x=y}^{\infty} \binom{x}{y} p^y (1-p)^{x-y} \cdot e^{-\lambda} \frac{\lambda^x}{x!} \quad \text{(Plug in Poisson and Binomial PMFs)} \\
&= e^{-\lambda} p^y \sum_{x=y}^{\infty} \frac{x!}{y!(x-y)!} (1-p)^{x-y} \frac{\lambda^x}{x!} \\
&= \frac{e^{-\lambda} p^y}{y!} \sum_{x=y}^{\infty} \frac{\lambda^x}{(x-y)!} (1-p)^{x-y} \\
&= \frac{e^{-\lambda} p^y}{y!} \sum_{k=0}^{\infty} \frac{\lambda^{k+y}}{k!} (1-p)^k \quad \text{(let } k = x - y\text{)} \\
&= \frac{e^{-\lambda} (\lambda p)^y}{y!} \sum_{k=0}^{\infty} \frac{(\lambda(1-p))^k}{k!} \\
&= \frac{e^{-\lambda} (\lambda p)^y}{y!} \cdot e^{\lambda(1-p)} \quad \text{(Taylor series for } e^{\lambda(1-p)}\text{)} \\
&= \frac{e^{-p\lambda} (\lambda p)^y}{y!}
\end{aligned} \tag{1}$$

So $Y \sim \text{Poisson}(p\lambda)$ and $\mathbb{E}[Y] = p\lambda$.

12. Variance of the geometric distribution

Independent trials each resulting in a success with probability p are successively performed. Let N be the time of the first success. Find the variance of N.

Solution:

Let $Y = 1$ if the first trial results in a success and $Y = 0$ otherwise. Now

$$\text{Var}(N) = \mathbb{E}[N^2] - (\mathbb{E}[N])^2$$

To calculate $\mathbb{E}[N^2]$, we condition on Y as follows:

$$\mathbb{E}[N^2] = \mathbb{E}[\mathbb{E}[N^2 | Y]]$$

However,

$$\mathbb{E}[N^2 | Y = 1] = 1$$

$$\mathbb{E}[N^2|Y=0] = \mathbb{E}[(1+N)^2]$$

These two equations follow because, if the first trial results in a success, then clearly $N = 1$ and so $N^2 = 1$. On the other hand, if the first trial results in a failure, then the total number of trials necessary for the first success will have the same distribution as one (the first trial that results in failure) plus the necessary number of additional trials. Since the latter quantity has the same distribution as N, we obtain that $\mathbb{E}[N^2|Y=0] = \mathbb{E}[(1+N)^2]$. Hence we see that

$$\begin{aligned}
\mathbb{E}[N^2] &= \mathbb{E}[N^2|Y=1]\mathbb{P}(Y=1) + \mathbb{E}[N^2|Y=0]\mathbb{P}(Y=0) \\
&= p + (1-p)\mathbb{E}[(1+N)^2] \\
&= 1 + (1-p)\mathbb{E}[2N + N^2]
\end{aligned}$$

Since we know that the expectation of a geometric random variable is given as $\mathbb{E}[N] = \frac{1}{p}$, by the Linearity of Expectation, we then have that

$$\begin{aligned}
\mathbb{E}[N^2] &= 1 + 2(1-p)\mathbb{E}[N] + (1-p)\mathbb{E}[N^2] \\
&= 1 + \frac{2(1-p)}{p} + (1-p)\mathbb{E}[N^2] \\
\mathbb{E}[N^2] - (1-p)\mathbb{E}[N^2] &= \frac{2-p}{p} \\
\mathbb{E}[N^2] &= \frac{2-p}{p^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{Var}(N) &= \mathbb{E}[N^2] - (\mathbb{E}[N])^2 \\
&= \frac{2-p}{p^2} - \frac{1}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}$$

13. 3 points on a line

(This problem uses the continuous law of total probability which has not yet be covered in class.) Three points X_1, X_2, X_3 are selected at random on a line L (continuous independent uniform distributions). What is the probability that X_2 lies between X_1 and X_3?

Solution:

Let $X_1, X_2, X_3 \sim Unif(0,1)$.

$$\begin{aligned}
\mathbb{P}(X_1 < X_2 < X_3) &= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < X_2 < X_3 \mid X_2 = x)\; f_{X_2}(x)\; dx && \text{Continuous LoTP} \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x, X_3 > x)\; f_{X_2}(x)\; dx && \text{Independence of } X_1, X_2, X_3 \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x)\; \mathbb{P}(x < X_3)\; f_{X_2}(x)\; dx && \text{Independence of } X_1, X_3 \\
&= \int_{-\infty}^{\infty} F_{X_1}(x)\; (1 - F_{X_3}(x))\; f_{X_2}(x)\; dx \\
&= \int_0^1 x\; (1-x)\; 1\; dx \\
&= \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{6}
\end{aligned}$$