CSE 312

Foundations of Computing II

Lecture 13: The Poisson Distribution

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Quiz 2: out Nov 8 6pm
due Nov 9 11:59pm

Zoo of Discrete RVs! 🦍🐘🦁🐅🦓🐪🦒

$X \sim \text{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$

$$E[X] = \frac{a + b}{2}$$

$$Var(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \text{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$

$$E[X] = p$$

$$Var(X) = p(1 - p)$$

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E[X] = np$$

$$Var(X) = np(1 - p)$$

$X \sim \text{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$

$$E[X] = \frac{1}{p}$$

$$Var(X) = \frac{1 - p}{p^2}$$

$X \sim \text{NegBin}(r, p)$

$$P(X = k) = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r}$$

$$E[X] = \frac{r}{p}$$

$$Var(X) = \frac{r(1 - p)}{p^2}$$

$X \sim \text{HypGeo}(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

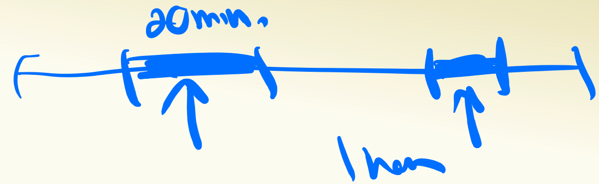
$$E[X] = n\frac{K}{N}$$

$$Var(X) = n\frac{K(N - K)(N - n)}{N^2(N - 1)}$$

Agenda

- Poisson Distribution
- Approximate Binomial distribution using Poisson distribution

Preview: Poisson



Model: # events that occur in an hour

– Expect to see 3 events per hour (but will be random)

– The expected number of events in t hours, is 3t

– Occurrence of events on disjoint time intervals is independent

Example – Model cars passing through a certain town in 1 hour

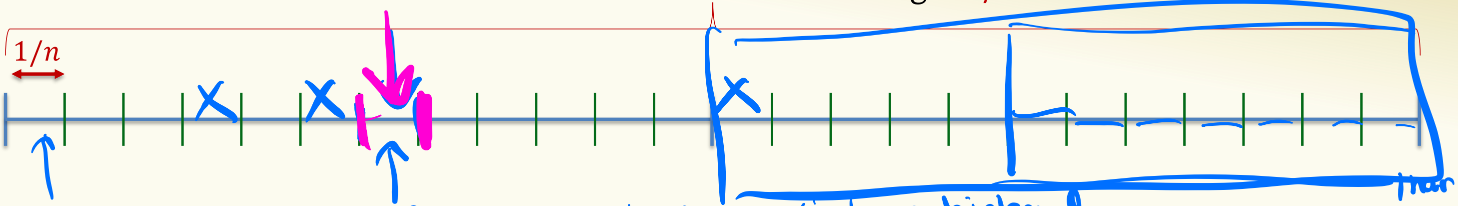
X = # cars passing through a certain town in 1 hour

$\frac{1}{2}$ hour $\quad Bin(\frac{n}{2}, \frac{3}{n}) =$ $\qquad \frac{n}{3} \cdot \frac{3}{n} = 1$

E #ev $= \frac{3}{2}$ $\qquad$ t hours

$\frac{1}{3}$ hour

Divide 1 hour into n intervals each of length $1/n$



p Prob an event in a particular subinterval of length $1/n$

Y : # events in hour

$Y \sim Bin(n, p)$

Bin

$E(X) = np = 3$

$p = \frac{3}{n}$

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What should p be?

Poll:

A. $\boxed{3/n}$
B. 3n
C. 3
D. 3/60

take limit as $n \to \infty$

$Bin(n, \frac{3}{n})$ → limiting r.v. X

Poisson

$$\Pr(X=0) = \lim_{n\to\infty} \underline{\Pr(Y=0)}$$

$$\lim_{x\to 0} \boxed{1-x \approx e^{-x}}$$

$$= \lim_{n\to\infty} \left(1-\frac{3}{n}\right)^n$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} \dots$$

$x = \frac{3}{n}$ $\quad 1-x \to e^{-x}$

$$= \left(e^{-\frac{3}{n}}\right)^n = e^{-3}$$

$Y \sim Bin(n, \frac{3}{n})$

$$\Pr(X=1) = \lim_{n\to\infty} \underbrace{\Pr(Y=1)}$$

$$= \lim_{n\to\infty} \underbrace{\binom{n}{1}\frac{3}{n}\left(1-\frac{3}{n}\right)^{n-1}}$$

$$\cancel{n}\cdot\frac{3}{\cancel{n}} \frac{\left(1-\frac{3}{n}\right)^n}{\left(1-\frac{3}{n}\right)}$$

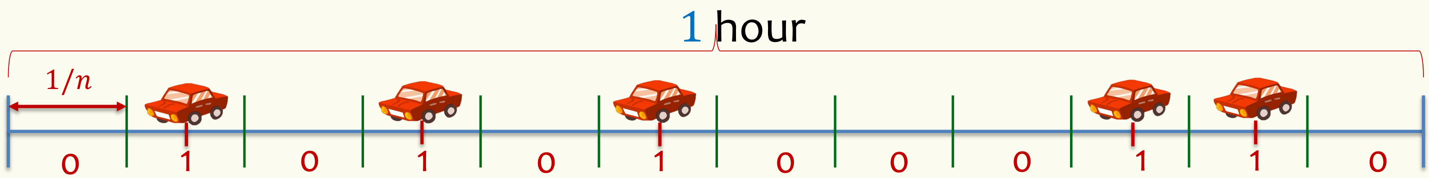
$$3\cdot e^{-3}$$

$$\Pr(X=k) = \lim_{n\to\infty} \Pr(Y=k) \binom{n}{k}\left(\frac{3}{n}\right)^k\left(1-\frac{3}{n}\right)^{n-k}$$

Example – Model the process of cars passing through a light in 1 hour

X = # cars passing through a light in 1 hour

Know: $\mathbb{E}(X) = \lambda$ for some given $\lambda > 0$



Discretize problem: n intervals, each of length $\frac{1}{n}$.

In each interval, a car passes by with probability $\frac{\lambda}{n}$ (assume ≤ 1 car can pass by)

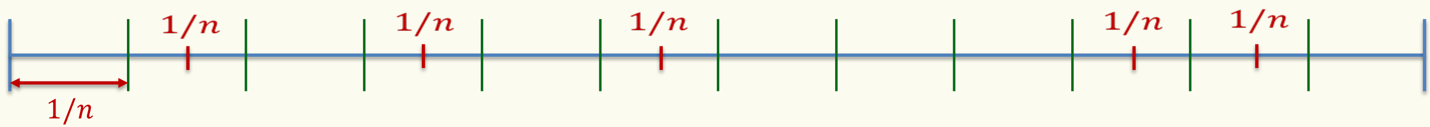
Bernoulli $X_i = 1$ if car in i-th interval (0 otherwise). $\mathbb{P}(X_i = 1) = \frac{\lambda}{n}$

$X = \sum_{i=1}^{n} X_i$ $\qquad$ $X \sim$ Binomial(n,p) $\qquad$ $\mathbb{P}(X = i) = \binom{n}{i}\left(\frac{\lambda}{n}\right)^{i}\left(1 - \frac{\lambda}{n}\right)^{n-i}$

indeed! $\mathbb{E}(X) = \lambda$

Don't like discretization

X is Binomial $\mathbb{P}(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}$



We want now $n \to \infty$

$$\mathbb{P}(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}=\underbrace{\frac{n!}{(n-i)!\,n^i}}_{\to 1}\frac{\lambda^i}{i!}\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}}\underbrace{\left(1-\frac{\lambda}{n}\right)^{-i}}_{\to 1}$$

$$\to \boxed{\mathbb{P}(X=i)=e^{-\lambda}\cdot\frac{\lambda^i}{i!}}$$

$\lambda = 3$

$i=0 \quad e^{-\lambda}$

$i=1 \quad e^{-\lambda}\lambda$

Poisson Distribution

Siméon Denis Poisson
1781-1840

- Suppose "events" happen, independently, at an *average* rate of λ per unit time.
- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. *with parameter* λ (denoted X ~ Poi(λ)) and has distribution (PMF):

$$\boxed{\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}}$$

$i = 0, 1, 2, \ldots$

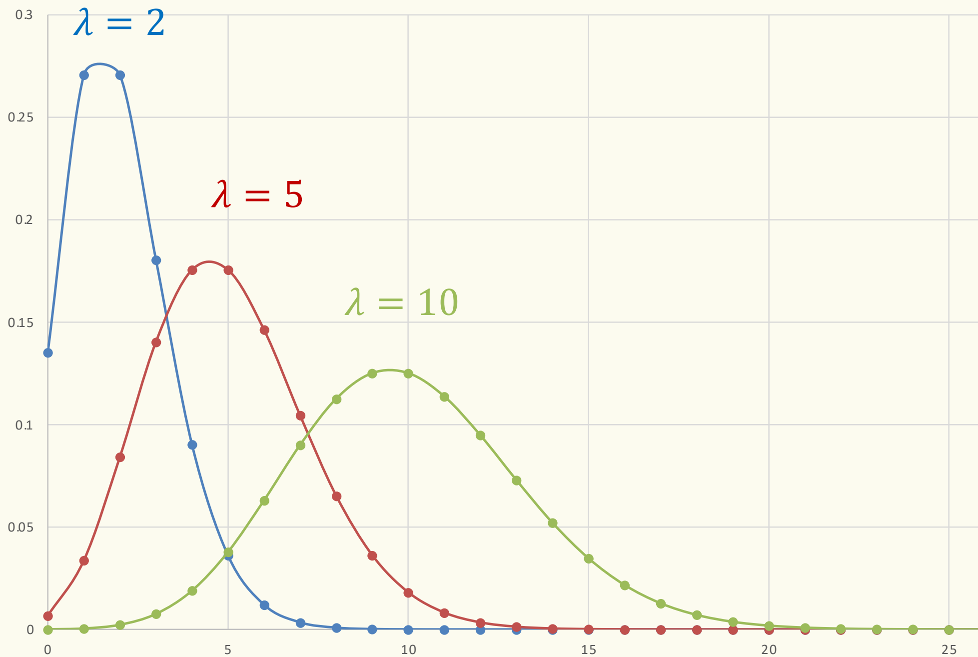
Several examples of "Poisson processes":

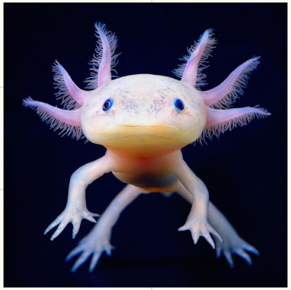
- # of cars passing through a certain town in 1 hour
- # of requests to web servers in a minute
- # of photons hitting a light detector in a given interval
- # of patients arriving to ER within an hour

Assume fixed average rate

Probability Mass Function

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$





Validity of Distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} \boxed{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}} = e^{-\lambda} e^{\lambda} = 1$$

$$= e^{\lambda}$$

Fact. $\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$

Validity of Distribution

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We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) = e^{-\lambda} \underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}} = e^{-\lambda} e^{\lambda} = 1$$

Fact. $\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$

Expectation

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then

$$\mathbb{E}(X) = \lambda$$

Proof. $\mathbb{E}(X) = \sum_{i=0}^{\infty} i \cdot \mathbb{P}(X = i) = \sum_{i=\cancel{0}\,1}^{\infty} i\, e^{-\lambda} \frac{\lambda^i}{i!}$

$$= e^{-\lambda} \sum_{i=1}^{\infty} \lambda^i \cdot \boxed{\frac{i}{i!}} = e^{-\lambda} \lambda \underbrace{\sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}}_{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}}$$

$$\frac{1}{(i-1)!}$$

$$= e^{-\lambda} \cdot \lambda\, e^{\lambda} = \lambda$$

Fact. $\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$

Expectation

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then

$$\mathbb{E}(X) = \lambda$$

Proof.

$$\mathbb{E}(X) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!}$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!}$$

= 1 (see prior slides!)

$$= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda$$

$Var(X) = E(X^2) - [E(X)]^2$

Variance

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then $\mathrm{Var}(X) = \lambda$

Proof. $\mathbb{E}(X^2) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 \quad = \lambda^2 + \lambda$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Variance

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then $\mathrm{Var}(X) = \lambda$

Proof. $$\mathbb{E}(X^2) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1)$$

$$= \lambda \left[\underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}(X) = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda$$

Similar to the previous proof
Verify offline.

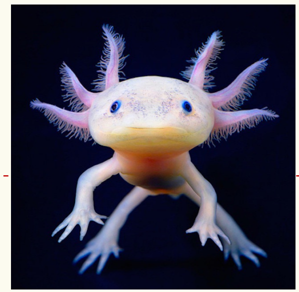
$$\Rightarrow \quad \mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

n $\frac{3}{n}$ 3.

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



Poisson approximates Binomial when n is very large, p is very small, and $\lambda = np$ is "moderate" (e.g. $n > 20$ and $p < 0.05$, $n > 100$ and $p < 0.1$)

Formally, Binomial is Poisson in the limit as $n \to \infty$ (equivalently, $p \to 0$) while holding $np = \lambda$

From Binomial to Poisson

$X \sim \text{Bin}(n, p)$

$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

$E[X] = np$

$Var(X) = np(1-p)$

$\boxed{n \to \infty}$

$np = \lambda$

$p = \frac{\lambda}{n} \to 0$

$\Rightarrow$

$X \sim \text{Poisson}(\lambda)$

$P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$E[X] = \lambda$

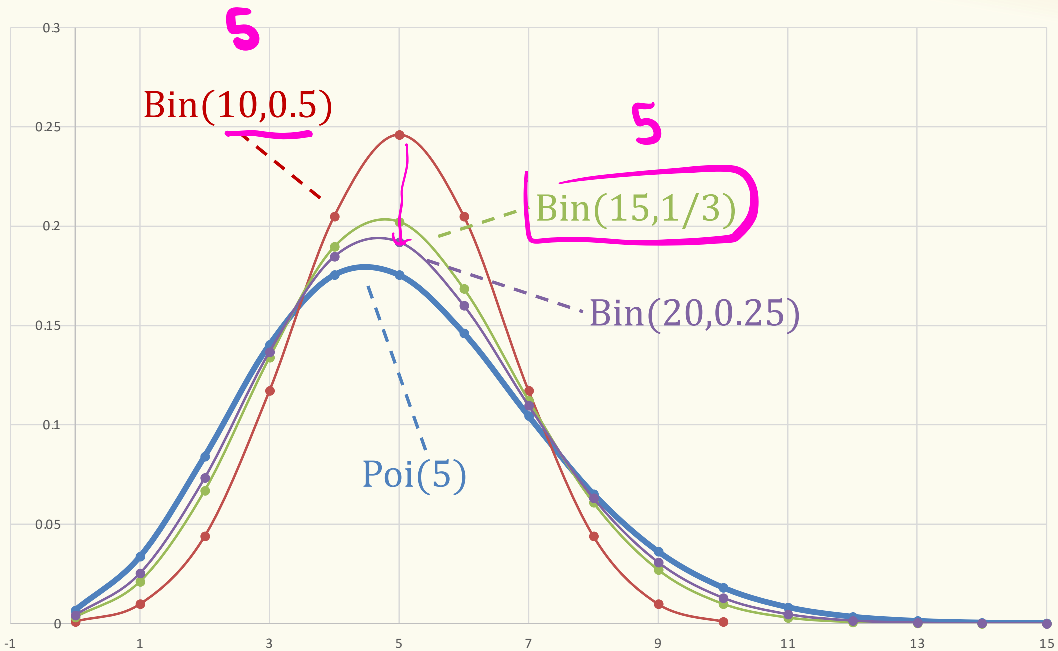
$Var(X) = \lambda$

Probability Mass Function – Convergence of Binomials

$\lambda = 5$

$p = \frac{5}{n}$

$n = 10,15,20$



$$as\ n \to \infty,\ \ Binomial(n,\ \ p = \ \lambda/n) \to poi(\lambda)$$

Example -- Approximate Binomial Using Poisson

Consider sending bit string over a network

- Send bit string of length **n = 10^4**
- Probability of (independent) bit corruption is **p = 10^{-6}**
- What is probability that message arrives uncorrupted?

corrupted bits Bin(10^4, 10^{-6})

Using Y ~ Bin(10^4, 10^{-6})

$$\mathbb{P}(Y = 0) = (1-10^{-6})^n$$

correct

Using X ~ Poi(λ = np = $10^4 \bullet 10^{-6}$ = 0.01)

$$\mathbb{P}(X = 0) = e^{-0.01}$$

approx.

Example -- Approximate Binomial Using Poisson

Consider sending bit string over a network

- Send bit string of length n = 10^4
- Probability of (independent) bit corruption is p = 10^{-6}
- What is probability that message arrives uncorrupted?

Using Y ~ Bin(10^4, 10^{-6})

$$\mathbb{P}(Y = 0) \approx 0.990049829$$

Using X ~ Poi(λ = np = $10^4 \bullet 10^{-6}$ = 0.01)

$$\mathbb{P}(X = 0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-0.01} \cdot \frac{0.01^0}{0!} = 0.990049834$$

Sum of Independent Poisson RVs

Theorem. Let $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$ such that $\lambda = \boxed{\lambda_1 + \lambda_2}$.
Let $Z = (X + Y)$. For all $k = 0,1,2,3 \ldots,$

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

More generally, let $X_1 \sim Poi(\lambda_1), \cdots, X_n \sim Poi(\lambda_n)$ such that $\lambda = \Sigma_i \lambda_i$.
Let $Z = \Sigma_i X_i$

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Sum of Independent Poisson RVs

Theorem. Let $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = (X + Y)$. For all $k = 0,1,2,3 \ldots,$

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

anything
P( ) = 0

0 1 2 k k-1 k-2...

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$\mathbb{P}(Z = k) = ?$

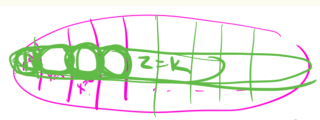
1. $\mathbb{P}(Z = k) = \Sigma_{j=0}^{k} \mathbb{P}(X = j, Y = k - j)$
2. $\mathbb{P}(Z = k) = \Sigma_{j=0}^{\infty} \mathbb{P}(X = j, Y = k - j)$
3. $\mathbb{P}(Z = k) = \Sigma_{j=0}^{k} \mathbb{P}(Y = k - j | X = j)\mathbb{P}(X = j)$
4. $\mathbb{P}(Z = k) = \Sigma_{j=0}^{k} \mathbb{P}(Y = k - j | X = j)$

Pr(Y=k-j)

Poll:
A. All of them are right
B. The first 3 are right
C. Only 1 is right
D. Don't know

LTP.

k=3



Pr(Z=3) = P(X=0, Y=3) + P(X=1, Y=2) + P(X=2, Y=1) + P(X=3, Y=0)

1. $\mathbb{P}(Z = k\ ) = \Sigma_{j=0}^{k}\ \mathbb{P}(X = j, Y = k - j)$
2. $\mathbb{P}(Z = k) = \Sigma_{j=0}^{\infty}\mathbb{P}(X = j, Y = k - j)$
3. $\mathbb{P}(Z = k) = \Sigma_{j=0}^{k}\mathbb{P}(Y = k - j|X = j)\mathbb{P}(X = j)$
4. $\mathbb{P}(Z = k) = \Sigma_{j=0}^{k}\mathbb{P}(Y = k - j|X = j)$

$\mathbb{P}(Z = k) = \Sigma_{j=0}^{k} \mathbb{P}(X = j, Y = k - j)$ Law of total probability

$= \Sigma_{j=0}^{k} \mathbb{P}(X = j)\mathbb{P}(Y = k - j)$ Independence

$$= \sum_{j=0}^{k} \frac{e^{-\lambda_1} \lambda_1^{j}}{j!} \frac{e^{-\lambda_2} \lambda_2^{k-j}}{(k-j)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \lambda_1^{j} \lambda_2^{k-j}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \boxed{(\lambda_1+\lambda_2)^k}$$

$$(a+b)^k = \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j}$$

λ_1 λ_2

Binomial Thm

Bin Thm

= pmf for Poisson$(\lambda_1+\lambda_2)$

$$\mathbb{P}(Z = z) = \Sigma_{j=0}^{k} \mathbb{P}(X = j, Y = z - j)$$

Law of total probability

$$= \Sigma_{j=0}^{k} \mathbb{P}(X = j)\mathbb{P}(Y = z - j) = \Sigma_{j=0}^{k} \; e^{-\lambda_1} \cdot \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{z-j}}{z - j!}$$

Independence

$$= e^{-\lambda} \left(\Sigma_{j=0}^{k} \; \cdot \frac{1}{j!\, z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right)$$

$$= e^{-\lambda} \left(\Sigma_{j=0}^{k} \; \frac{z!}{j!\, z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right) \frac{1}{z!}$$

$$= e^{-\lambda} \cdot (\lambda_1 + \lambda_2)^z \cdot \frac{1}{z!} = e^{-\lambda} \cdot \lambda^z \cdot \frac{1}{z!}$$

Binomial Theorem

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Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

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General principle:

- Events happen at an average rate of λ per time unit
- Number of events happening at a time unit X is distributed according to Poi(λ)
- Poisson approximates Binomial when n is large, p is small, and np is moderate
- Sum of independent Poisson is still a Poisson

Next Time

- Continuous Random Variables
- Probability Density Function
- Cumulative Density Function

Often we want to model experiments where the outcome is not discrete.