CSE 312

Foundations of Computing II

Lecture 14: Continuous Random Variables

PAUL G. ALLEN SCHOOL
OF COMPUTER SCIENCE & ENGINEERING

Anna R. Karlin

Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Agenda

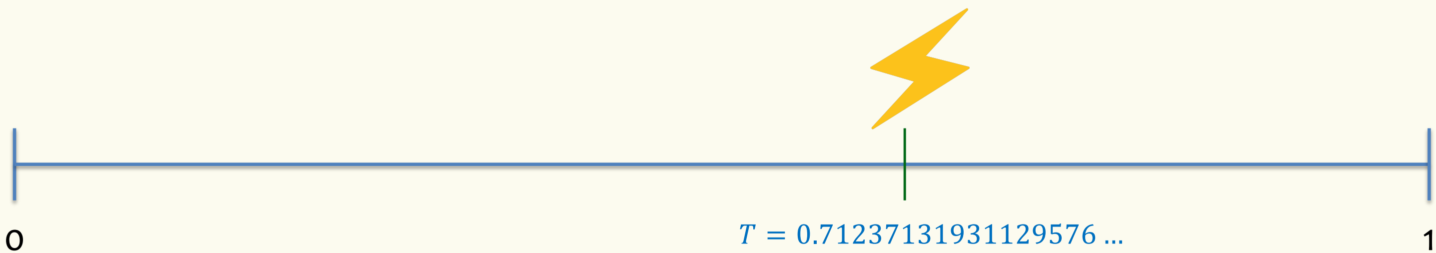
- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function

Often we want to model experiments where the outcome is <u>not</u> discrete.

Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every time within $[0,1]$ is equally likely
  - – Time measured with infinitesimal precision.

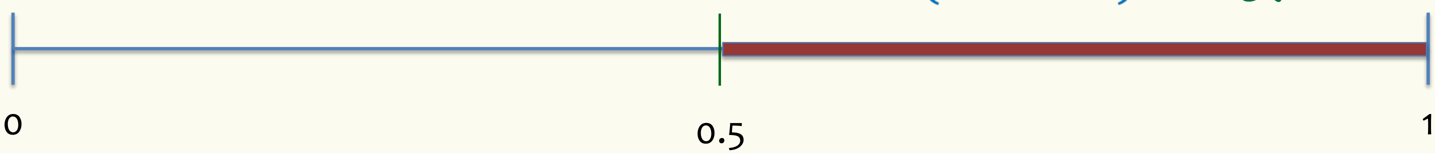


The outcome space is not discrete

Lightning strikes a pole within a one-minute time frame

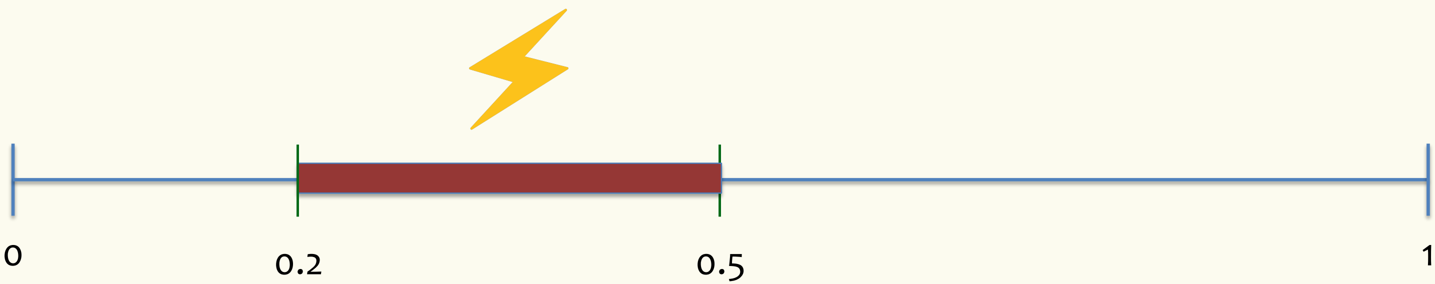
- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$$\mathbb{P}(T \geq 0.5) = \frac{1}{2}$$



Lightning strikes a pole within a one-minute time frame

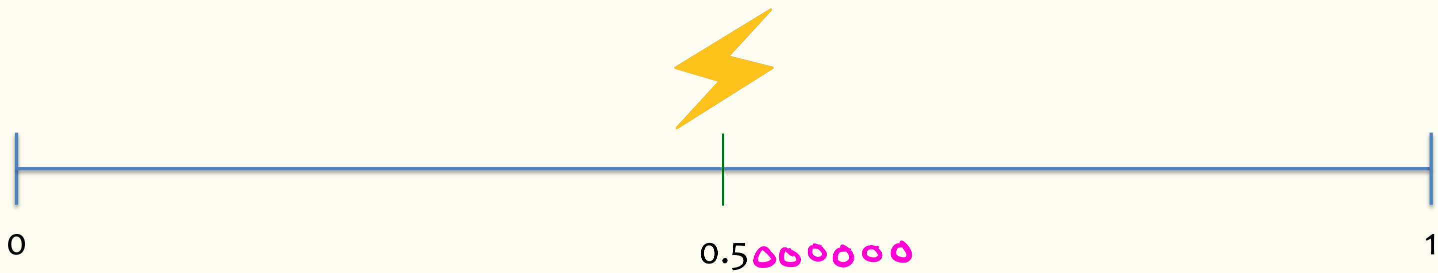
- T = time of lightning strike
- Every point in time within [0,1] is equally likely



$$\mathbb{P}(0.2 \leq T \leq 0.5) = 0.3$$

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely



$\mathbb{P}(T = 0.5) =$ 0

Bottom line

- This gives rise to a different type of random variable
- $\mathbb{P}(T = x) = 0$ for all $x \in [0,1]$
- Yet, somehow we want
  - $\mathbb{P}(T \in [0,1]) = 1$
  - $\mathbb{P}(T \in [a, b]) = b - a$ 	$0 < a < b < 1$
  - …
- How do we model the behavior of T?

- **Discrete Approximation?**

Discrete r.v. X with range $\Omega_X = \{0, 1, 2, \dots\}$

pmf

$p_X(x) = \Pr(X = x)$

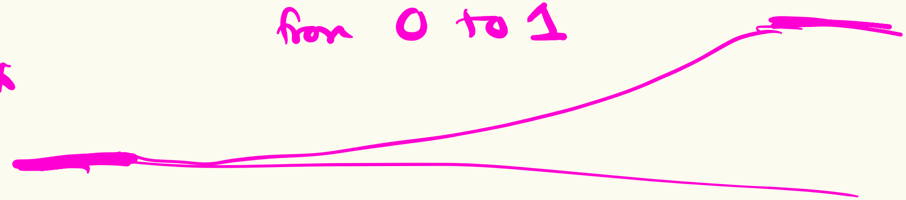
$$\sum_{x \in \Omega_X} p_X(x) = 1$$

$p_X(x) \geq 0 \quad \forall x$

CDF

$F_X(x) = \Pr(X \leq x)$

F_X is monotone increasing from 0 to 1



$$F_X(w) = \sum_{\substack{x \in \Omega_X \\ \text{s.t. } x \leq w}} p_X(x)$$

Poll: Given the CDF, how do you compute the pmf?

https://pollev.com/ annakarlin185

$\Pr(X = k) = p_X(k) =$

a. $F_X(k-1)$

b. $F_X(1) + F_X(2) + \cdots + F_X(k-1)$

c. $\boxed{F_X(k) - F_X(k-1)}$

d. I don't know.

$0 \quad \frac{1}{2} \quad 1 \quad \frac{3}{2} \quad 2$

$p_X(k) = \Pr(X=k) = F_X(k) - F_X(k-1)$

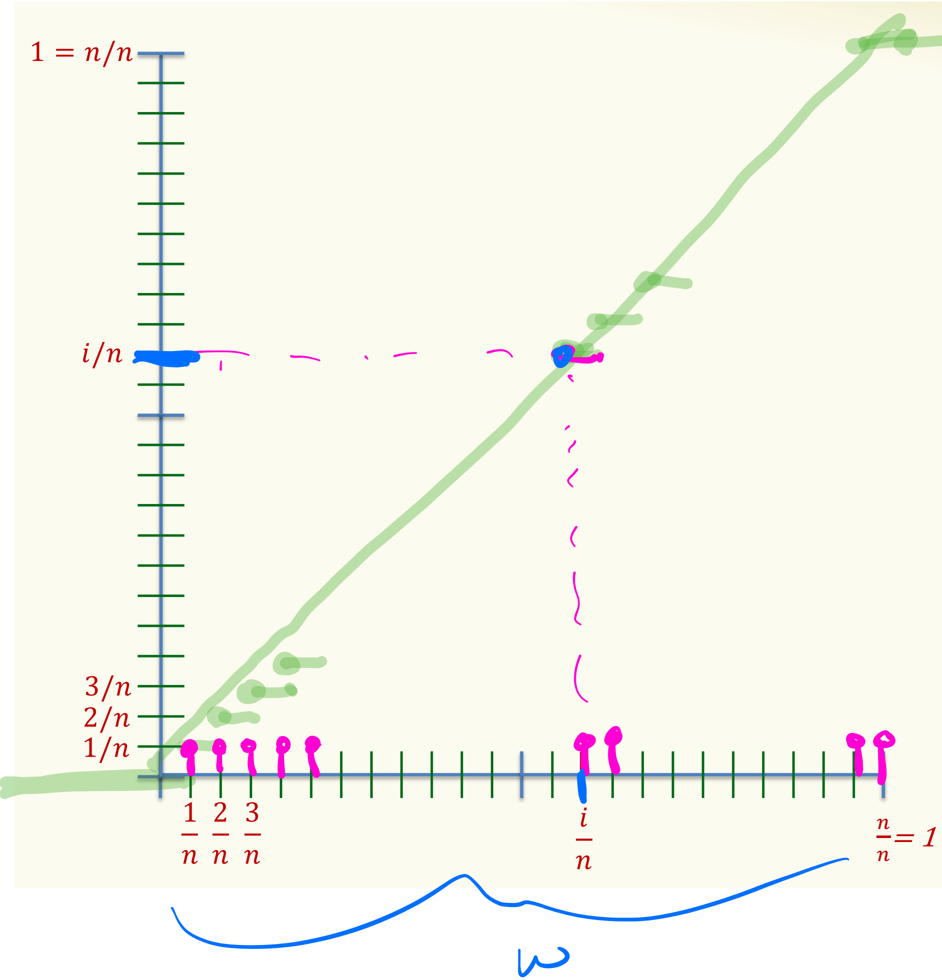
$F_X(w) = \sum_{x \in \Omega_X \text{ s.t. } x \le w} p_X(x)$

Want to represent cont random var.

uniform random draw $[0,1]$

$\Omega_X = \{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}\}$

$\Pr(X=x) = \begin{cases} \ldots & 1 \le j \le n \\ 0 & \text{elsewhere} \end{cases}$



$F_X(\omega) = \begin{cases} 0 & \omega \le 0 \\ \frac{j}{n} & \frac{j}{n} \le \omega < \frac{j+1}{n},\ j = 0 \ldots n-1 \\ 1 & \omega \ge 1 \end{cases}$

$\lim_{n\to\infty} F_X(\omega) = \begin{cases} 0 & \omega \le 0 \\ \omega & \\ 1 & \omega \ge 1 \end{cases}$

$\lim_{n\to\infty} P(X=x) = 0$

pmf no longer makes sense

introduce probability density fn

$p_X(k) = \Pr(X=k) = \frac{F_X(k) - F_X(k-1)}{k-(k-1)}$

$F_X(w) = \sum_{\substack{x \in \Omega_X \\ \text{s.t. } x \le w}} p_X(x)$

$f_X(x) = \frac{d}{dx} F_X(x)$

pdf.

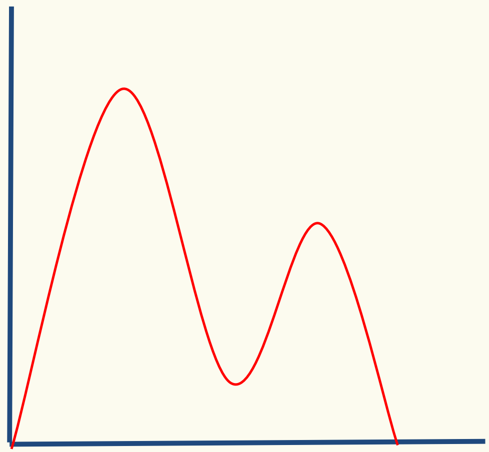
$F_X(w) = \int_{-\infty}^{w} f_X(x)\,dx$

$f_X(x) = 1$

$0 \le x \le 1$

$F_X(w) = w$

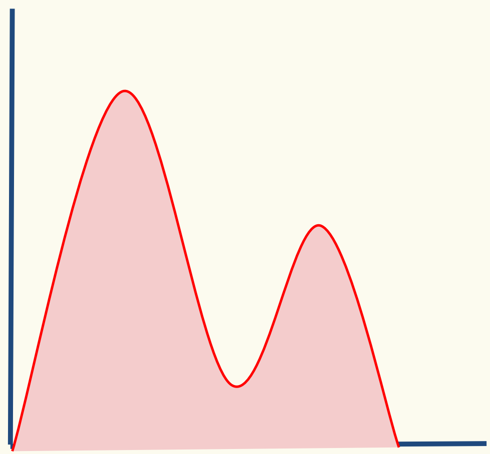
Definition. A **continuous random variable*** X is defined by a **probability density function** (PDF) $f_X : \mathbb{R} \to \mathbb{R}$, such that



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

$$\sum_{x \in \Omega_X} p_X(x) = 1$$

Probability Density Function - Intuition

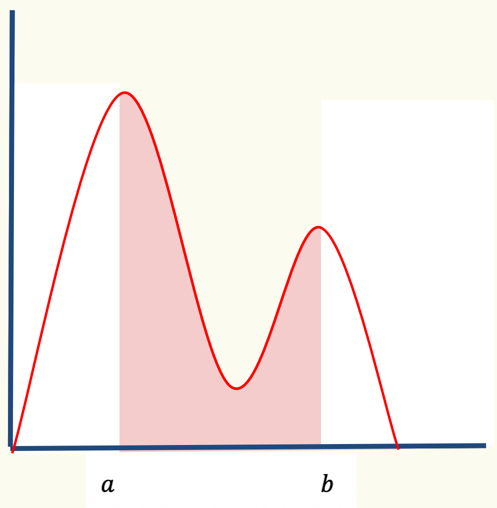


Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$\Pr(a \le X \le b) = \sum_{\substack{\omega \in \Omega_X \\ a \le \omega \le b}} p_X(\omega)$$

Probability Density Function - Intuition

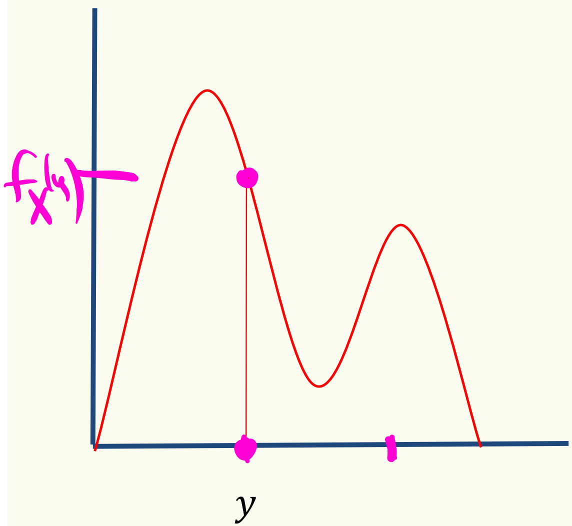


Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\, \mathrm{d}x = 1$

$$\underline{P(a \leq X \leq b)} = \int_a^b f_X(x)\, \mathrm{d}x$$

Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

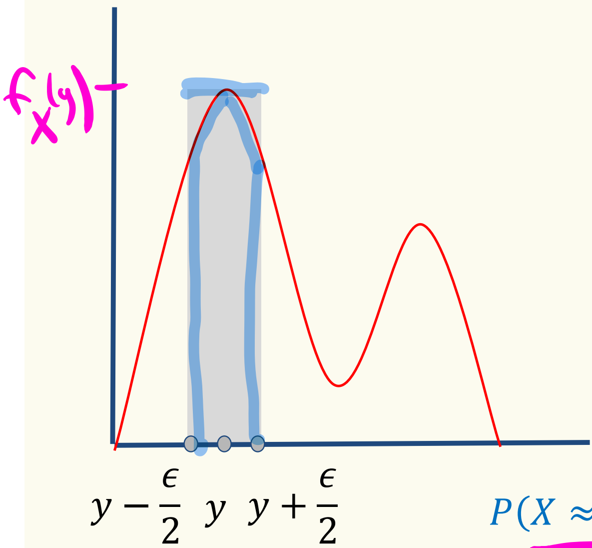
$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

Density $\neq$ Probability

$f_X(y) \neq 0$ $\quad$ $\mathbb{P}(X = y) = 0$

Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

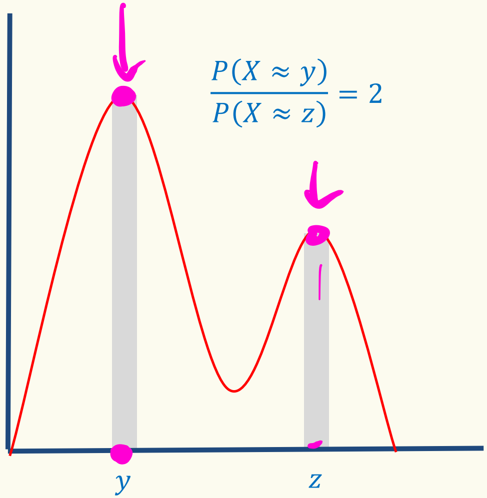
Normalization: $\int_{-\infty}^{+\infty} f_X(x)\, \mathrm{d}x = 1$

$$\boxed{P(a \leq X \leq b) = \int_a^b f_X(x)\, \mathrm{d}x}$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\, \mathrm{d}x = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\, \mathrm{d}x \approx \epsilon f_X(y)$$

Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\cancel{\epsilon} f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X : \mathbb{R} \to \mathbb{R}$, such that

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

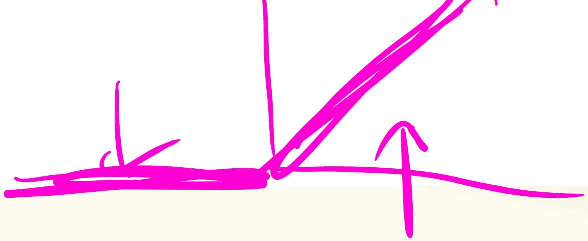
Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$



$$F_X(x) = \begin{cases} 0 & x \le 0 \\ x & x \in [0,1] \\ 1 & x \geq 1 \end{cases}$$

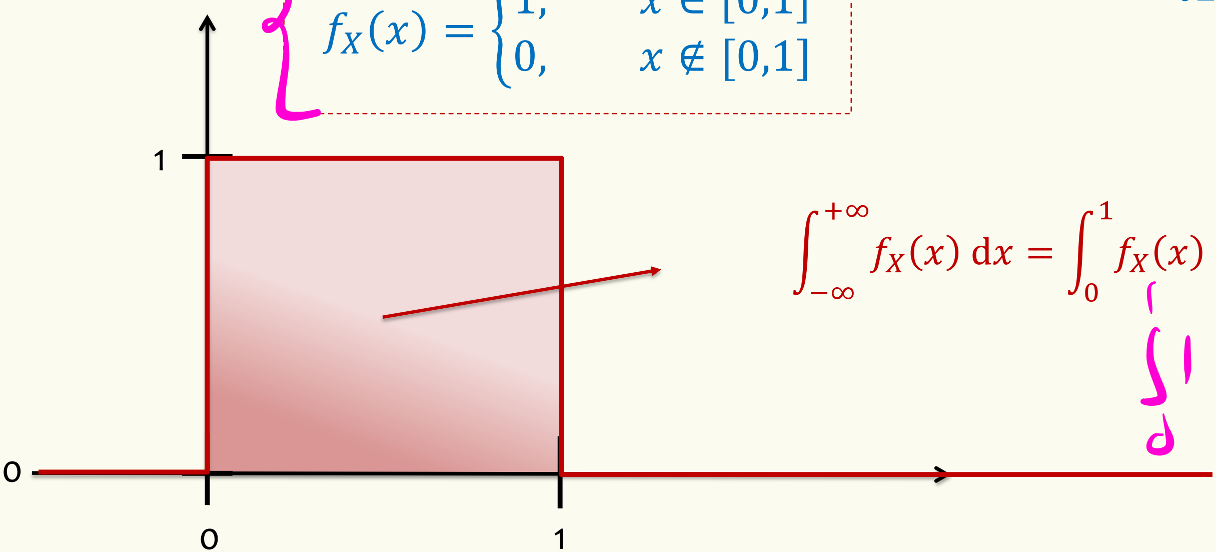
PDF of Uniform RV

$X \sim \mathrm{Unif}(0,1)$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



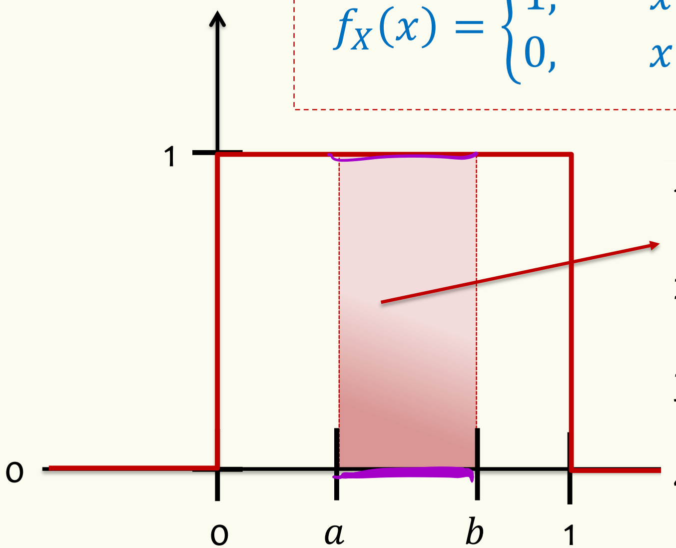
$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 f_X(x)\,\mathrm{d}x = 1 \cdot 1 = 1$$

$$\int_0^1 1\,dx = 1$$

Probability of Event

$X \sim \text{Unif}(0,1)$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



1. If $0 \leq a$ *and* $b \leq 1$
   $\mathbb{P}(a \leq X \leq b) = b - a$
2. If $a < 0$ *and* $0 \leq b \leq 1$
   $\mathbb{P}(a \leq X \leq b) = b$
3. If $a \geq 0$ *and* $b > 1$
   $\mathbb{P}(a \leq X \leq b) = b - a$
4. If $a < 0$ *and* $b > 1$
   $\mathbb{P}(a \leq X \leq b) = 1$

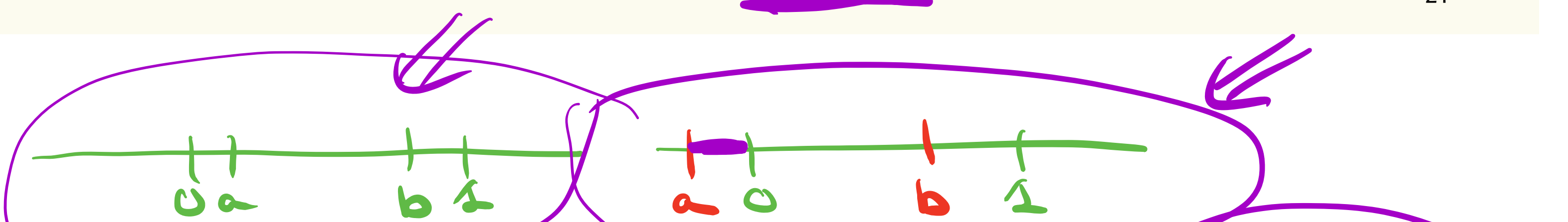
Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

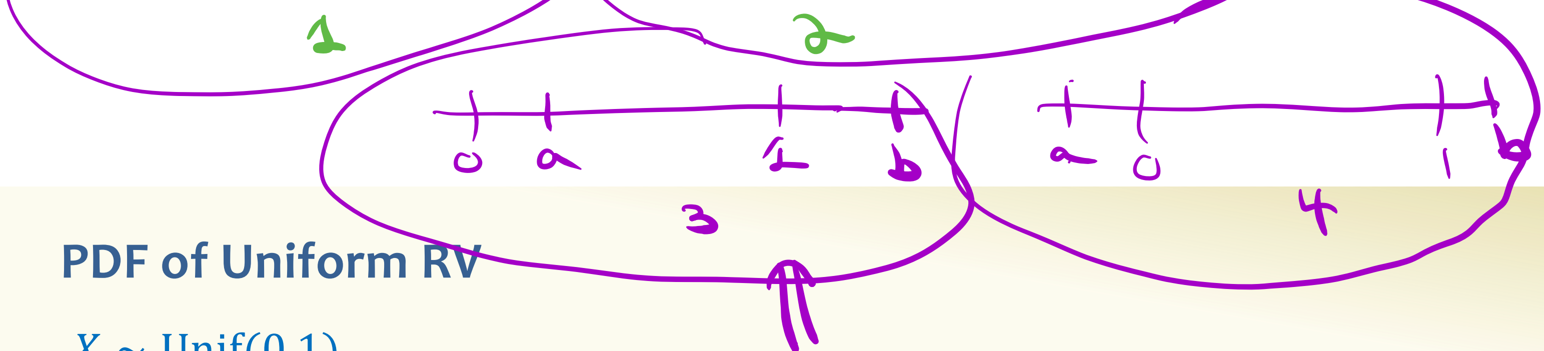
Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$\boxed{P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x}$$

pollev/ annakarlin185

A. All of them are correct
B. Only 1, 2, 4 are right
C. Only 1 is right
D. Only 1 and 2 are right

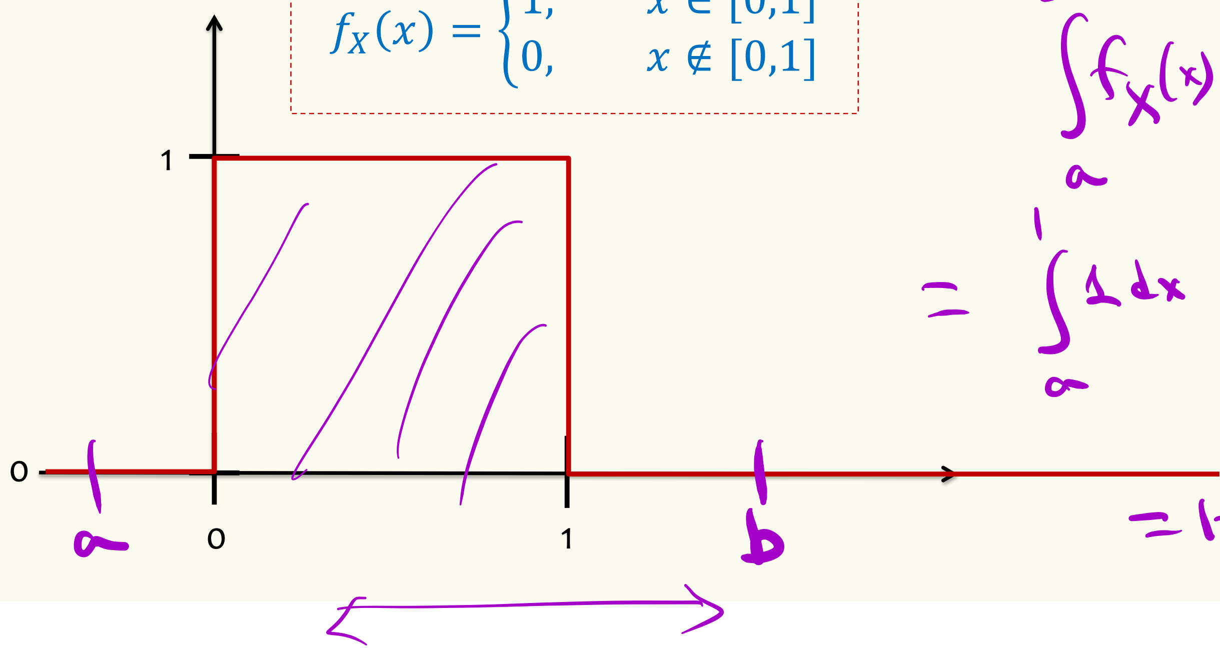




PDF of Uniform RV

$X \sim \mathrm{Unif}(0,1)$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



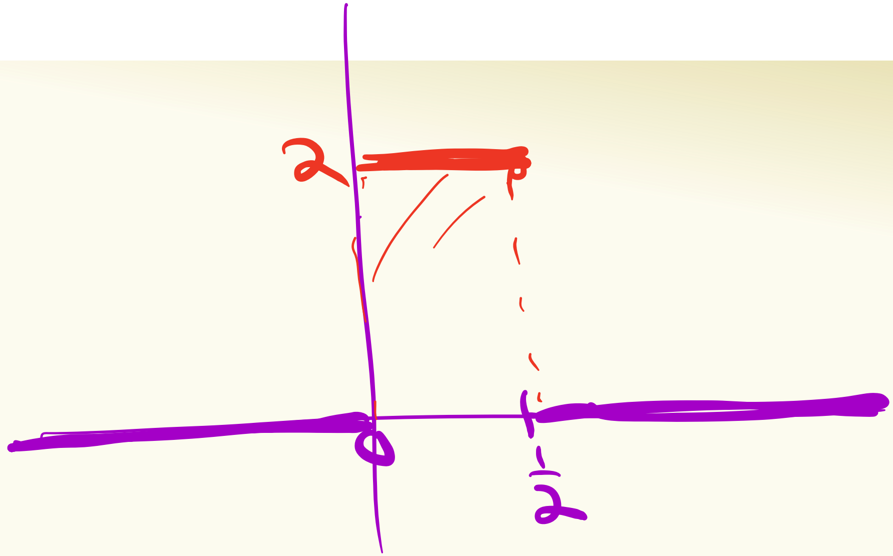
$$\int_a^b f_X(x)\,dx$$

$$= \int_a^1 1\,dx + \int_1^b 0\,dx$$

$$= 1 - a$$

$\mathrm{Unif}[0, \tfrac{1}{2}]$

$f_X(x)$



$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

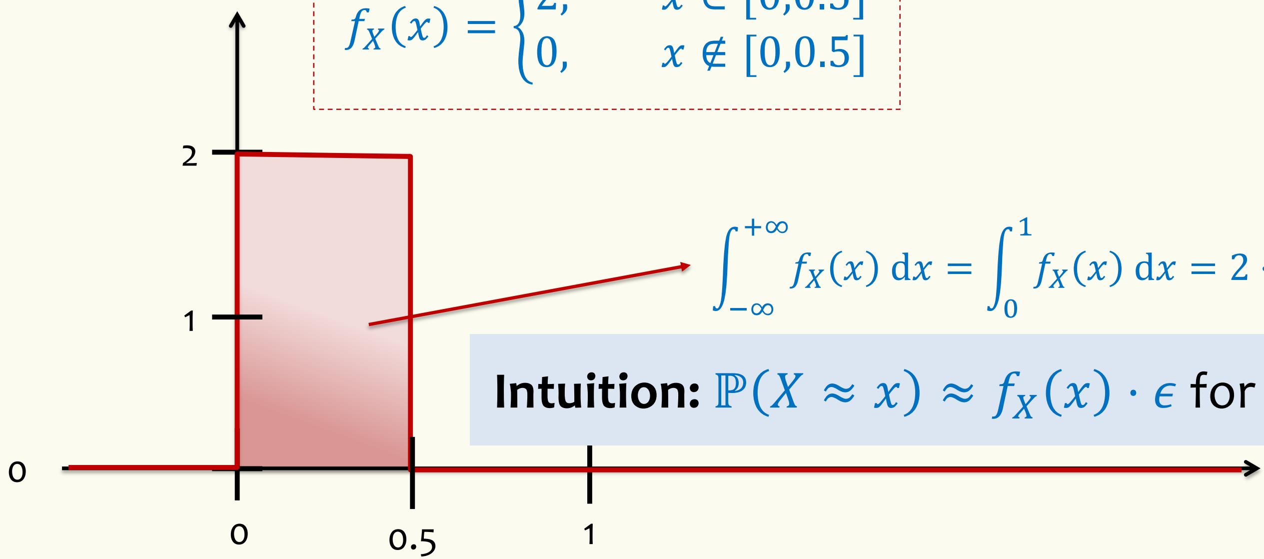
PDF of Uniform RV

$X \sim \text{Unif}(0,0.5)$

Density $\neq$ Probability

$f_X(x) \gg 1$ is possible!

$$f_X(x) = \begin{cases} 2, & x \in [0,0.5] \\ 0, & x \notin [0,0.5] \end{cases}$$



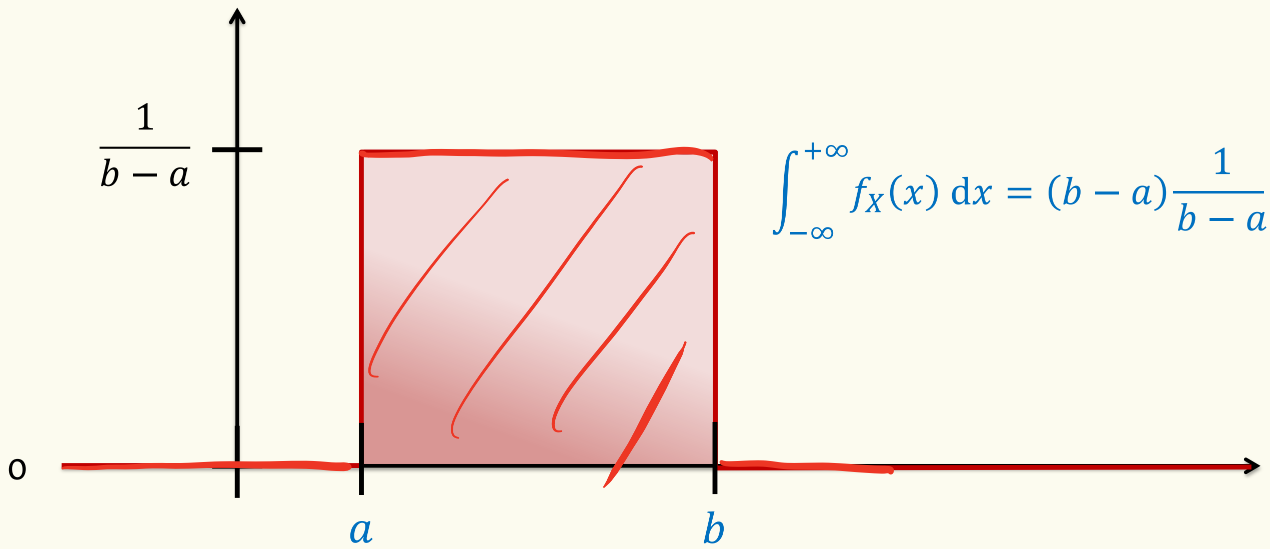
$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 f_X(x)\,\mathrm{d}x = 2 \cdot 0.5 = 1$$

Intuition: $\mathbb{P}(X \approx x) \approx f_X(x) \cdot \epsilon$ for small ϵ

Uniform Distribution

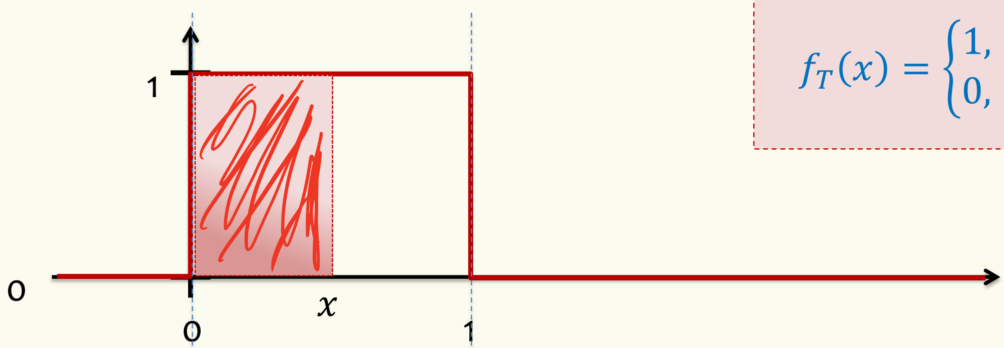
$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$X \sim \text{Unif}(a, b)$



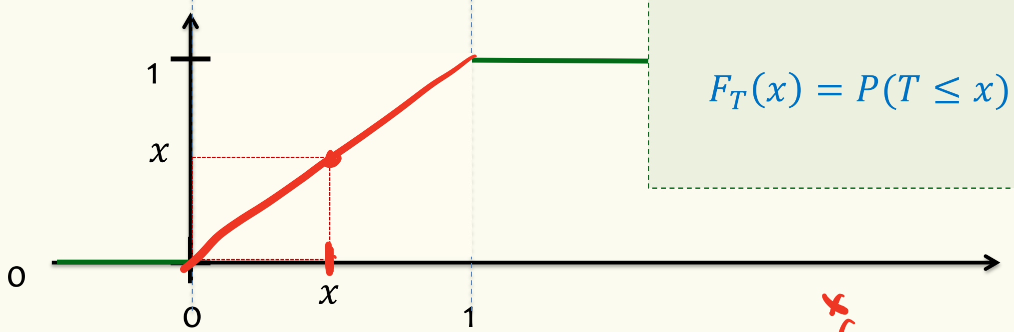
$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = (b-a)\frac{1}{b-a} = 1$$

Example. $T \sim \mathrm{Unif}(0,1)$



Probability Density Function

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



Cumulative Distribution Function

$$F_T(x) = P(T \le x) = \begin{cases} 0 & x \le 0 \\ \cancel{x} & 0 \le x \le 1 \\ 1 & 1 \le x \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(w)\,dw$$

$x \in [0,1)$ $= \int_0^x 1 \, dw = x$

Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = \mathbb{P}(X \leq a) = \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F(x)$

Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = \underbrace{\mathbb{P}(X \leq a)}_{\text{CDF}} = \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F(x)$

Therefore: $\mathbb{P}(X \in [a, b]) = F(b) - F(a)$

F_X is monotone increasing, since $f_X(x) \geq 0$. That is $F_X(c) \leq F_X(d)$ for $c \leq d$

$\mathrm{Lim}_{a\to-\infty} F_X(a) = P(X \leq -\infty) = 0$ $\quad$ $\mathrm{Lim}_{a\to+\infty} F_X(a) = P(X \leq +\infty) = 1$

From Discrete to Continuous

$p_X(x) \geq 0$ $\quad\quad\quad\quad$ $f_X(x)$

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

LOTUS

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$$\int_{-\infty}^{\infty} x^2 \underbrace{f_X(x)}_{\text{pdf}}\,dx$$

Expectation of a Continuous RV

$E(X) = \sum_{x \in \Omega_X} x \, \underbrace{\Pr(X=x)}_{p_X(x)}$

Definition. The **expected value** of a continuous RV X is defined as

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Fact. $\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$

Definition. The **variance** of a continuous RV X is defined as

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot \left(x - \mathbb{E}(X)\right)^2 \mathrm{d}x = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

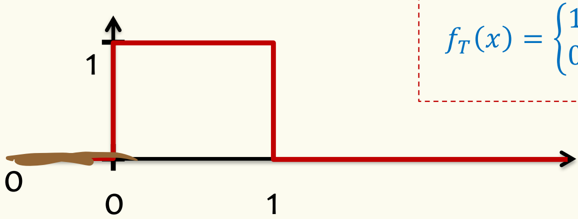
$\mathrm{Var}(X) = E\left(\left(X - E(X)\right)^2\right)$

Expectation of a Continuous RV

Definition.

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Example. $T \sim \mathrm{Unif}(0,1)$



$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

$f_X(x)$

$$E(X) = \int_0^1 1 \cdot x \, dx$$

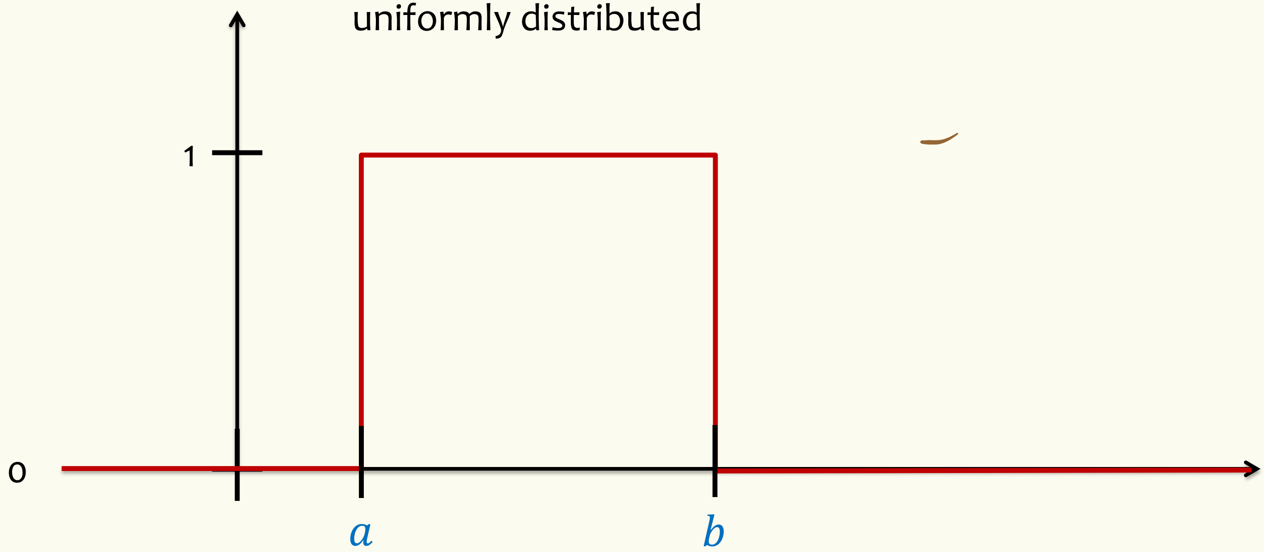
$$= \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

Uniform Distribution

$X \sim \text{Unif}(a, b)$

We also say that X follows the uniform distribution / is uniformly distributed

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



Uniform Density – Expectation

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \mathrm{Unif}(a, b)$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Uniform Density – Expectation

$X \sim \mathrm{Unif}(a, b)$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}(X) = \boxed{\int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x}$$

$$= \frac{1}{b-a} \int_a^b x \, \mathrm{d}x = \frac{1}{b-a} \left(\frac{x^2}{2} \right) \Bigg|_a^b = \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right)$$

$$= \frac{(b-a)(a+b)}{2(b-a)} = \frac{a+b}{2}$$

x^k

$\dfrac{x^{k+1}}{k+1}$

Uniform Density – Variance

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \mathrm{Unif}(a, b)$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, \mathrm{d}x$$

$E(X^2)$

Uniform Density – Variance

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \text{Unif}(a,b)$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x)\, x^2 \,\mathrm{d}x$$

$$= \frac{1}{b-a}\int_a^b x^2 \,\mathrm{d}x = \frac{1}{b-a}\left(\frac{x^3}{3}\right)\Bigg|_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

Uniform Density – Variance

$$\mathbb{E}(X^2) = \frac{b^2 + ab + a^2}{3} \qquad \mathbb{E}(X) = \frac{a+b}{2}$$

$X \sim \mathrm{Unif}(a, b)$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
&= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12} \\
&= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}
\end{aligned}$$