CSE 312

Foundations of Computing II

Lecture 15: Exponential and Normal Distribution

PAUL G. ALLEN SCHOOL
OF COMPUTER SCIENCE & ENGINEERING

Anna R. Karlin

Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Quiz: Out Monday, Nov 8 6pm
Due Tuesday, Nov 9 11:59pm

Discrete: $p_X(x) = Pr(X=x)$ $\longleftrightarrow$ $F_X(x) = Pr(X \le x)$

Review – Continuous RVs

Probability Density Function (PDF).

$f: \mathbb{R} \to \mathbb{R}$ s.t.

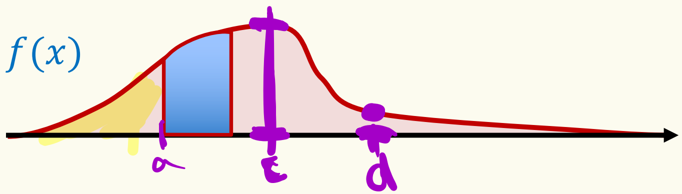
- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1$

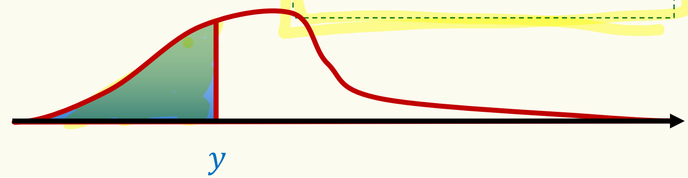
Cumulative Dist'n ~~Density~~ Function (CDF).

$$\boxed{F(y) = \int_{-\infty}^{y} f(x)\,\mathrm{d}x} = Pr(X \le y)$$

$$\boxed{\textbf{Theorem. } f(x) = \frac{dF(x)}{dx}}$$

$f(x)$





Density ≠ Probability !

$$\boxed{\begin{aligned}\mathbb{P}(X \in [a,b]) &= \int_a^b f_X(x)\mathrm{d}x \\ &= F_X(b) - F_X(a)\end{aligned}}$$

$$F(y) = \mathbb{P}(X \leq y)$$

$$= P(X < y)$$

$E(X) = \sum_{x \in \Omega_X} x \Pr(X=x)$

sum → int

pmf → pdf

Expectation of a Continuous RV

Definition. The **expected value** of a continuous RV X is defined as

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Fact. $\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$

Definition. The **variance** of a continuous RV X is defined as

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot \left(x - \mathbb{E}(X)\right)^2 \mathrm{d}x = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$\mathrm{Var}(X) = E\left[(X - E(X))^2\right] = E(X^2) - E(X)^2$

$$E(g(X)) = \int_{-\infty} g(x) f(x)\, dx$$

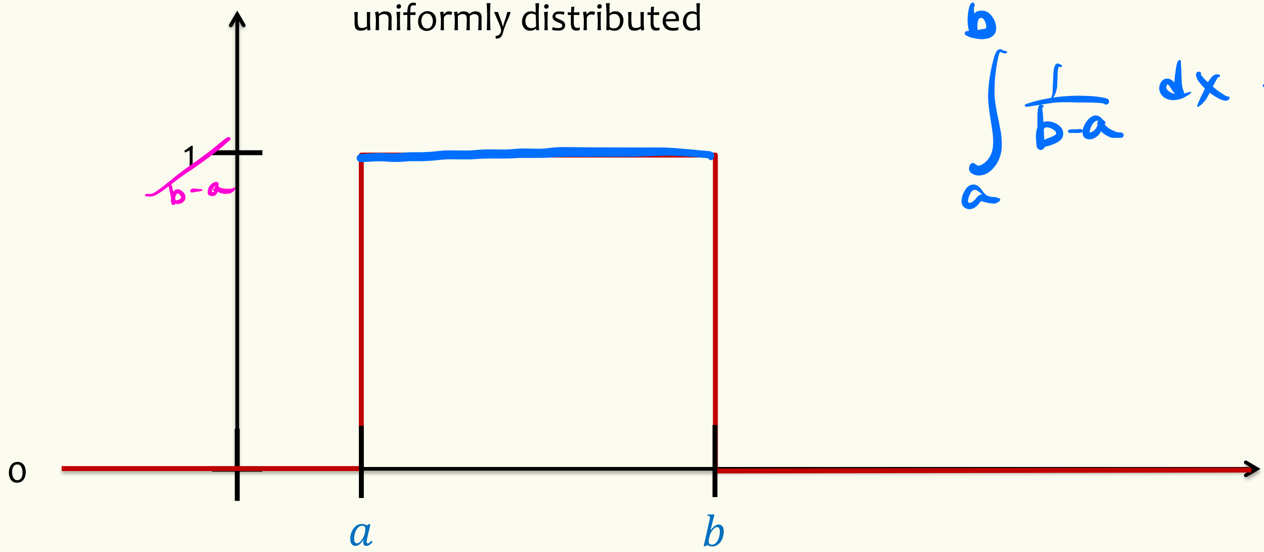
Uniform Distribution

$X \sim \mathrm{Unif}(a, b)$

We also say that X follows the uniform distribution / is uniformly distributed

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\int_a^b \frac{1}{b-a}\, dx = 1$$



Uniform Density – Expectation

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \mathrm{Unif}(a,b)$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \,\mathrm{d}x$$

$$= \frac{1}{b-a}\int_a^b x\,\mathrm{d}x = \frac{1}{b-a}\left(\frac{x^2}{2}\right)\Bigg|_a^b = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right)$$

$$= \frac{(b-a)(a+b)}{2(b-a)} = \frac{a+b}{2}$$

Uniform Density – Variance

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \mathrm{Unif}(a,b)$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x)\, x^2 \,\mathrm{d}x$$

$$= \frac{1}{b-a}\int_a^b x^2\,\mathrm{d}x = \frac{1}{b-a}\left(\frac{x^3}{3}\right)\Bigg|_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

$E(X^2) - \left(E(X)\right)^2$

$\frac{a+b}{2}$

Uniform Density – Variance

$$\mathbb{E}(X^2) = \frac{b^2 + ab + a^2}{3} \qquad \mathbb{E}(X) = \frac{a + b}{2}$$

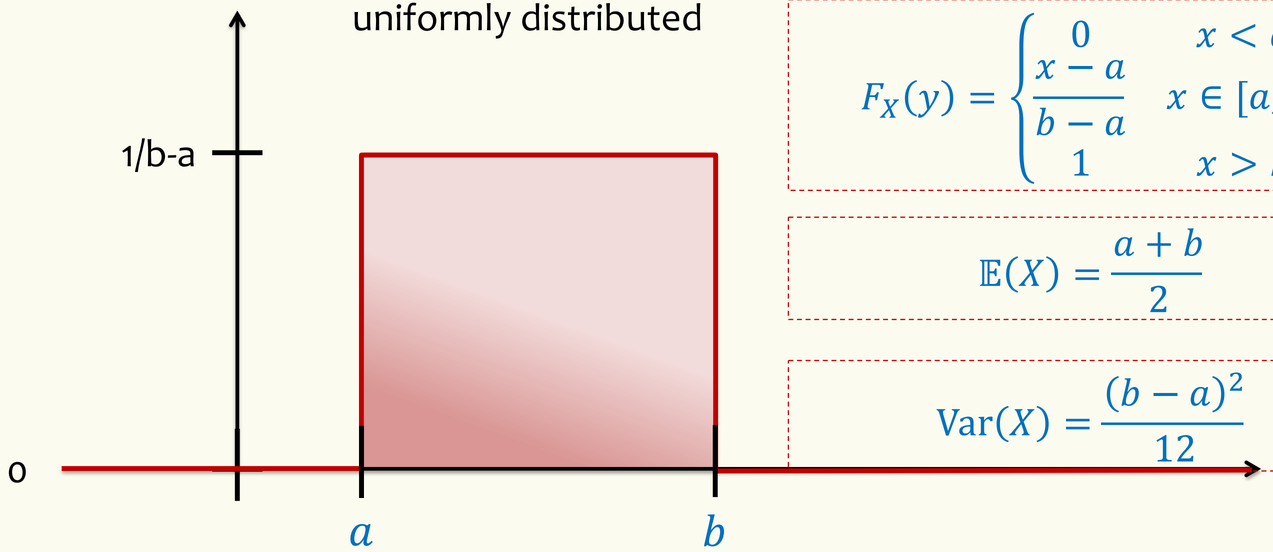
$X \sim \text{Unif}(a, b)$

$$\begin{aligned}
\text{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
&= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12} \\
&= \frac{b^2 - 2ab + a^2}{12} = \frac{(b - a)^2}{12}
\end{aligned}$$

Uniform Distribution

$X \sim \mathrm{Unif}(a, b)$

We also say that X follows the uniform distribution / is uniformly distributed



$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$F_X(y) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$

$$\mathbb{E}(X) = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

0

Exponential Density

Assume expected # of occurrences of an event per unit of time is λ

- Cars going through intersection
- Number of lightning strikes
- Requests to web server
- Patients admitted to ER

Numbers of occurrences of event: Poisson distribution

$$\boxed{\mathbb{P}(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}}$$

(Discrete) $i = 0, 1, 2, \ldots$

How long to wait until next event? <u>Exponential density!</u>

Let's define it and then derive it!

The Exponential PDF/CDF

1 unit

Assume expected # of occurrences of an event per unit of time is λ

Numbers of occurrences of event: Poisson distribution Poisson(λ)

How long to wait until next event? Exponential density!

- The exponential RV has range $[0, \infty]$, unlike Poisson with range $\{0,1,2,\ldots\}$
- Let $Y \sim Exp(\lambda)$ be the time till the first event. We will compute $F_Y(t)$ and $f_Y(t)$

$$Pr(Y > t) = Pr(X = 0) \underset{\text{Poisson}}{=} \boxed{e^{-\lambda t}}$$

$$\boxed{\begin{array}{c} Pr(Y \le t) \\ F_Y(t) \end{array}} = 1 - e^{-\lambda t} \qquad t \geq 0$$

0 t

X: # events in t units of time $\sim$ Poisson$(t\lambda)$

$$\Rightarrow f_Y(t) = \frac{d}{dt}F_Y(t) = \lambda e^{-\lambda t}$$

$t < 0$

$Pr(Y \leq t) = 0$

The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is λ

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

- The exponential RV has range $[0, \infty]$, unlike Poisson with range $\{0,1,2,\ldots\}$
- Let $Y \sim Exp(\lambda)$ be the time till the first event. We will compute $F_Y(t)$ and $f_Y(t)$
- Let $X \sim Poi(t\lambda)$ be the # of events in the first t units of time, for $t \geq 0$.
- $P(Y > t) = P(no\ event\ in\ the\ first\ t\ units) = P(X = 0) = e^{-t\lambda}\frac{t\lambda^0}{0!} = e^{-t\lambda}$
- $F_Y(t) = 1 - P(Y > t) = 1 - e^{-t\lambda}$
- $f_Y(t) = \frac{d}{dt}F_Y(t) = \lambda e^{-t\lambda}$

Exponential Distribution

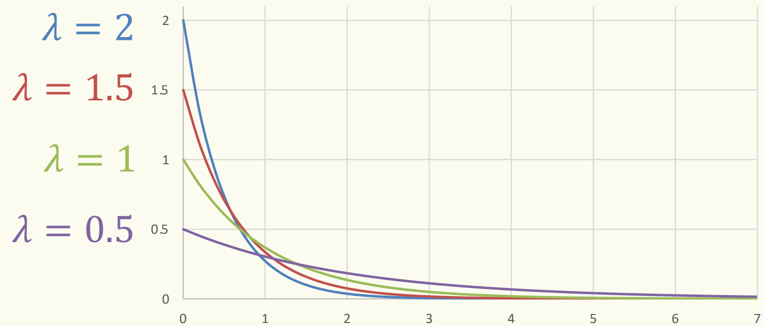
Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \mathrm{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,
$F_X(y) = 1 - e^{-\lambda y}$

$F_X(y) = 0 \quad y < 0$



Expectation

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

$$= \int_{0}^{\infty} \lambda e^{-\lambda x} x \, dx$$

Expectation

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \,\mathrm{d}x$$

$$= \int_0^{+\infty} \lambda e^{-\lambda x} \cdot x \,\mathrm{d}x$$

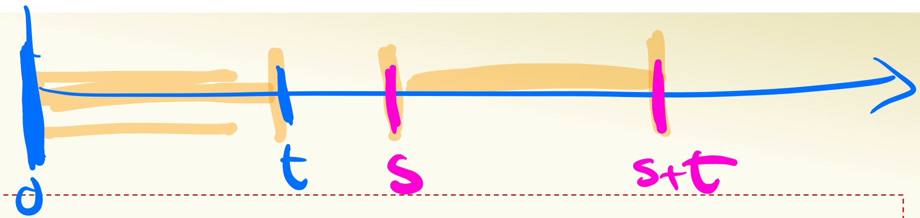
$$= \left(-(x + \frac{1}{\lambda})e^{-\lambda x}\right)\Big|_0^{\infty} = \frac{1}{\lambda}$$

Somewhat complex calculation
use integral by parts

$$\mathbb{E}(X) = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Memorylessness



Definition. A random variable is **memoryless** if for all $s, t > 0$,

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

Fact. $X \sim \mathrm{Exp}(\lambda)$ is memoryless.

Assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as $s = 0$

Memorylessness of Exponential

Assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as $s = 0$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Proof.

$$\mathbb{P}(X > s + t \mid X > s) = \frac{\Pr(X > s+t, X > s)}{P(X > s)}$$

$$= \frac{\Pr(X > s+t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = \Pr(X > t)$$

$$= \Pr(X > t)$$

$\Pr(X > a) = e^{-\lambda a}$

$X \sim \exp(\lambda)$

Memorylessness of Exponential

Assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as $s = 0$

Fact. $X \sim \mathrm{Exp}(\lambda)$ is memoryless.

Proof.

$$\mathbb{P}(X > s + t \mid X > s) = \frac{\mathbb{P}(\{X > s + t\} \cap \{X > s\})}{\mathbb{P}(X > s)}$$

$$= \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(X > t)$$

The only memoryless RVs are the geometric RV (discrete) and Exp RV (continuous)

$X \sim \exp(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$\lambda = \frac{1}{10}$

$E(X) = \frac{1}{\lambda}$

example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins.

λ

$T \sim \exp(\frac{1}{10})$

$$\Pr(10 \leq T \leq 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}} dx$$

$$= F_T(20) - F_T(10) = (1 - e^{-\frac{20}{10}}) - (1 - e^{-\frac{10}{10}})$$

$= e^{-1} - e^{-2}$

example

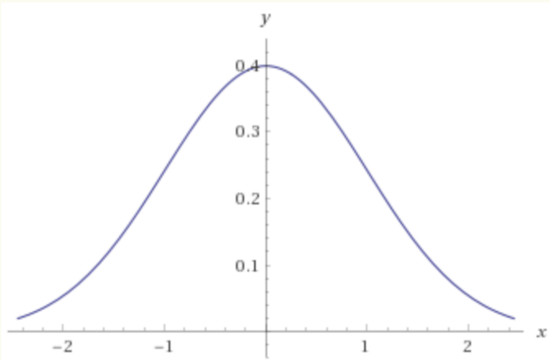
- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins.

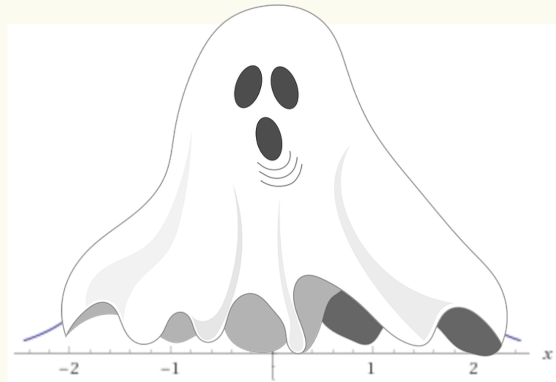
$$T \sim Exp\left(\frac{1}{10}\right)$$

$$P(10 \le T \le 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}} \, dx$$

$$y = \frac{x}{10}, dy = \frac{dx}{10}$$

$$P(10 \le T \le 20) = \int_{1}^{2} e^{-y} \, dy = -e^{-y} \Big|_{1}^{2} = e^{-1} - e^{-2}$$





Normal Distribution

Paranormal Distribution

The Normal Distribution

Definition. A **Gaussian (or <u>normal</u>) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$)

mean variance,

Carl Friedrich Gauss

The Normal Distribution

Carl Friedrich Gauss

Definition. A **Gaussian (or <u>normal</u>) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$)

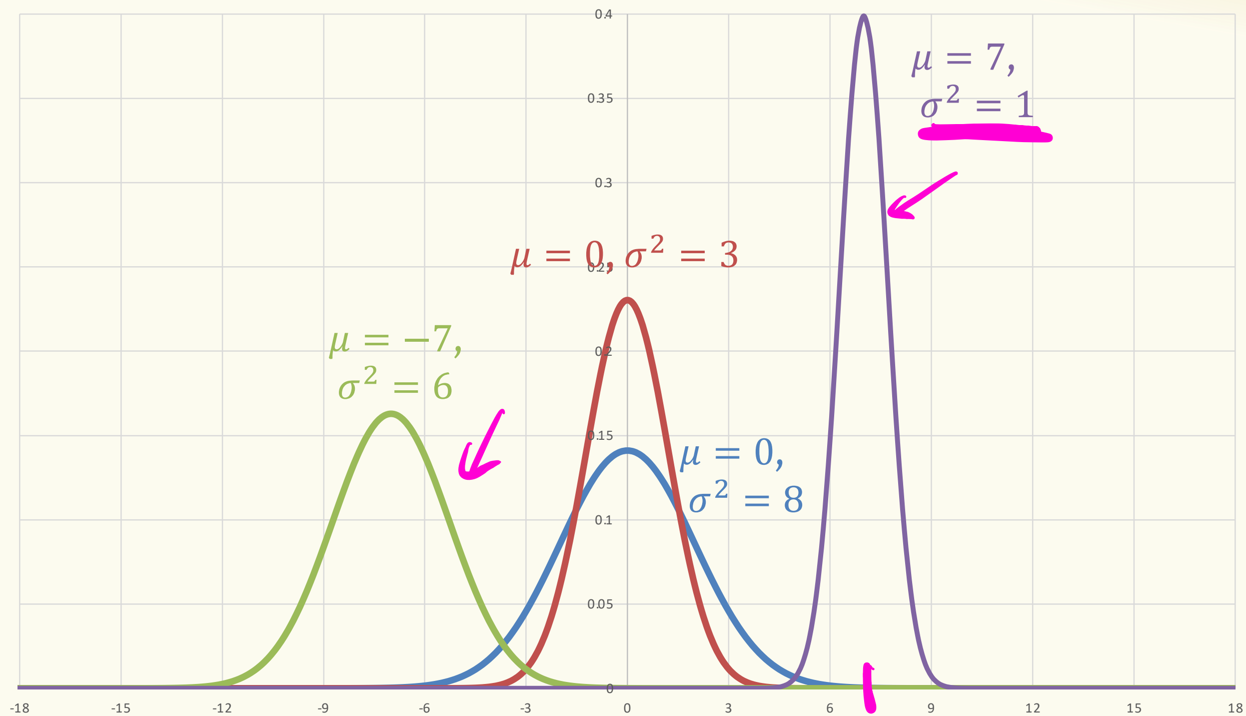
Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}(X) = \mu$, and $\mathrm{Var}(X) = \sigma^2$

Expectation follows from density being symmetric around μ, $f_X(\mu - x) = f_X(\mu + x)$

We will see next time why the normal distribution is (in some sense) the most important distribution.

The Normal Distribution

Aka a "Bell Curve" (imprecise name)



Shifting and Scaling the Normal Distribution

Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = aX + b$

What is $\mathbb{E}(Y) = E(aX+b) = aE(X) + b = a\mu + b$

$\mathrm{Var}(Y) = \mathrm{Var}(aX+b) = \mathrm{Var}(aX) = a^2\mathrm{Var}(X)$

$= a^2\sigma^2$

What is mean and variance of $Z = \boxed{\dfrac{X-\mu}{\sigma}}$ **?**

$a = \frac{1}{\sigma}$ $\quad b = -\frac{\mu}{\sigma}$

$E(Z) = \frac{1}{\sigma}\cdot\mu - \frac{\mu}{\sigma} = 0$

$\mathrm{Var}(Z) = a^2\sigma^2 = 1$

Z has
Exp 0
Var 1

Closure of normal distribution – Under Shifting and Scaling

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

We know:

$$\mathbb{E}(Y) = a\ \mathbb{E}(X) + b = a\mu + b$$

$$\mathrm{Var}(Y) = a^2\ \mathrm{Var}(X) = a^{2}\sigma^2$$

Note: $\boxed{\frac{X-\mu}{\sigma}} \sim \mathcal{N}(0, 1)$

μ exp σ^2 variance σ std deviation

Closure of the normal -- under addition

Fact. If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent normal RV) then $aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

Note: The special thing is that the sum of normal RVs is still a normal RV.
The values of the expectation and variance is not surprising.

X $-X$