CSE 312

Foundations of Computing II

Lecture 2: Permutations, combinations, the Binomial Theorem and more.

PAUL G. ALLEN SCHOOL
OF COMPUTER SCIENCE & ENGINEERING

Anna R. Karlin

Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Agenda (very unlikely we will get through all of this)

- Recap & Examples
- Binomial Theorem
- Multinomial Coefficients
- Combinatorial Proofs
- Inclusion-Exclusion
- Pigeonhole Principle
- Stars and Bars

Quick Summary

- Sum Rule

  If you can choose from

  – Either one of n options,

  – OR one of m options with NO overlap with the previous n,

  then the number of possible outcomes of the experiment is $n + m$

- Product Rule

  In a sequential process, if there are

  - n_1 choices for the first step,
  - n_2 choices for the second step (given the first choice), …, and
  - n_k choices for the k^{th} step (given the previous choices),

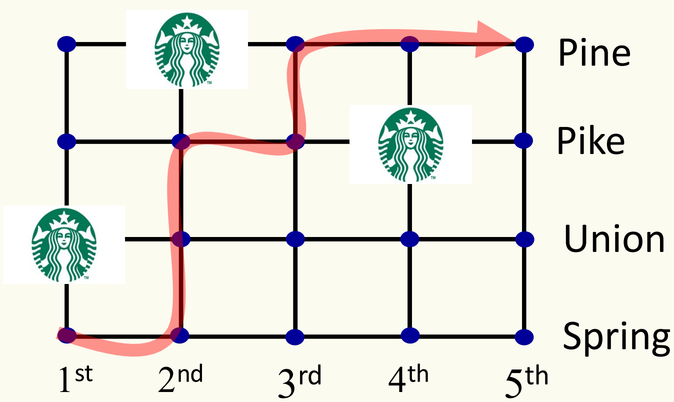
  then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_k$

- Complementary Counting

Quick Summary

- K-sequences: How many length k sequences over alphabet of size n? repetition allowed.
  - Product rule ➔ n^K
- K-permutations: How many length k sequences over alphabet of size n, without repetition?
  - Permutation ➔ $\frac{n!}{(n-k)!}$
- K-combinations: How many size k subsets of a set of size n (without repetition and without order)?
  - Combination ➔ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

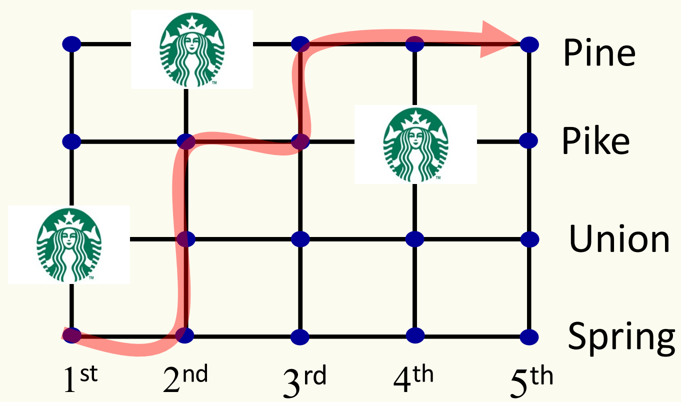
Example – Counting Paths



"How many ways to walk from 1st and Spring to 5th and Pine only going ↑ and → ?

_____ _____ _____ _____ _____ _____ _____

Example – Counting Paths -2



"How many ways to walk from 1st and Spring to 5th and Pine only going ↑ and → ?

Poll:

A. 2^7

B. $\frac{7!}{4!}$

C. $\binom{7}{4} = \frac{7!}{4!3!}$

D. $\binom{7}{3} = \frac{7!}{3!4!}$

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Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Proof. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

Why??

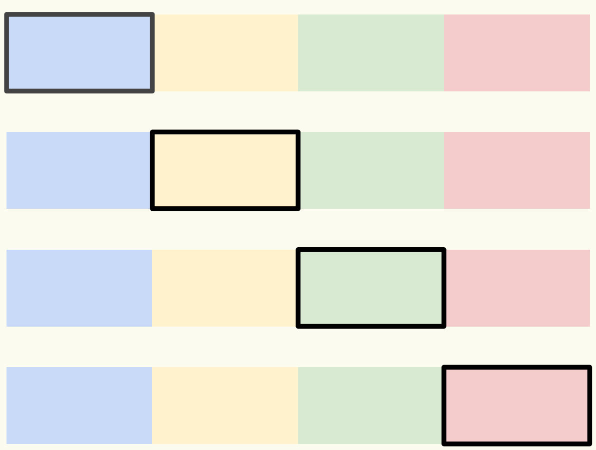
This is called an Algebraic proof, i.e., Prove by checking algebra

Symmetry in Binomial Coefficients – A different proof

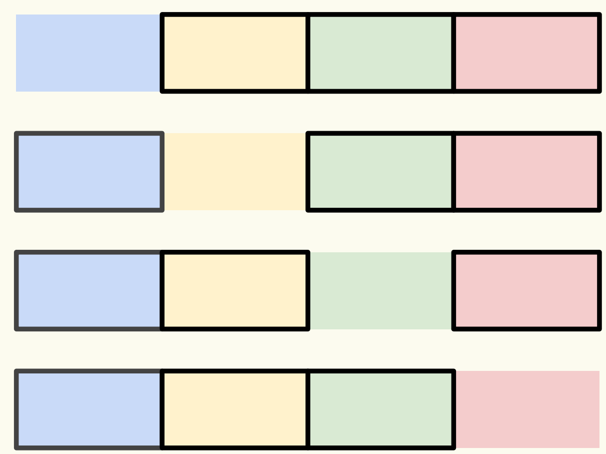
Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which $n - k$ elements are excluded



$$\binom{\mathbf{4}}{\mathbf{1}} = \mathbf{4} = \binom{\mathbf{4}}{\mathbf{3}}$$



Symmetry in Binomial Coefficients – A different proof

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

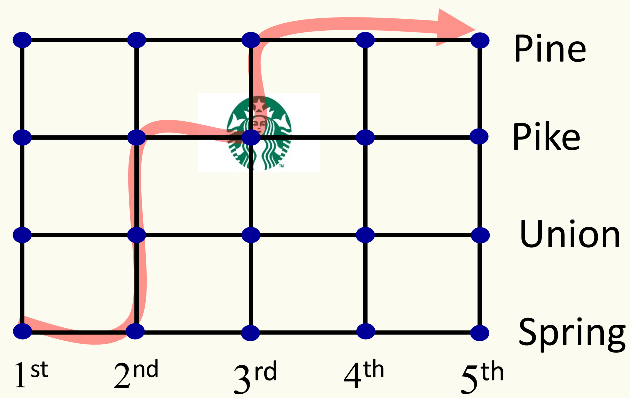
1. Choose which k elements are included
2. Choose which $n-k$ elements are excluded

This is called a combinatorial argument/proof

- Let S be a set of objects
- Show how to count $|S|$ one way => $|S| = N$
- Show how to count $|S|$ another way => $|S| = m$

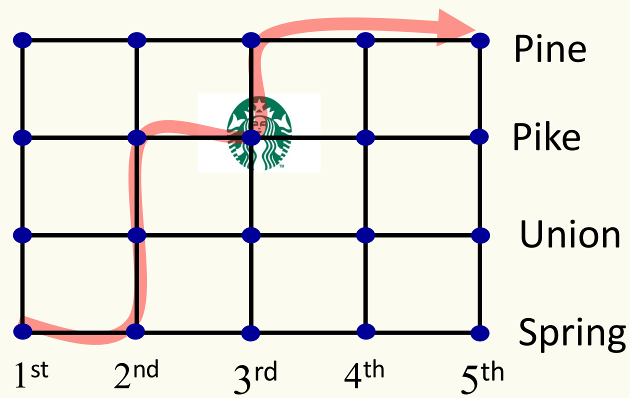
More examples of combinatorial proofs coming soon!

Example – Counting Paths - 3



"How many ways to walk from 1st and Spring to 5th and Pine only going ↑ and → but stopping at Starbucks on 3rd and Pike?"

Example – Counting Paths - 3



"How many ways to walk from 1st and Spring to 5th and Pine only going ↑ and → but stopping at Starbucks on 3rd and Pike?"

Poll:

A. $\binom{7}{3}$

B. $\binom{7}{3}\binom{7}{1}$

C. $\binom{4}{2}\binom{3}{1}$

D. $\binom{4}{2}\binom{3}{2}$

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Agenda

- Recap & Examples
- Binomial Theorem
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Binomial Theorem: Idea

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= xx + xy + yx + yy\\ &= x^2 + 2xy + y^2\end{aligned}$$

$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)\,(x+y)\,(x+y)\\ &= xxxx + yyyy + xyxy + yxyy + \ \dots\end{aligned}$$

Binomial Theorem: Idea

Poll: What is the coefficient for xy^3?

A. 4

B. $\binom{4}{1}$

C. $\binom{4}{3}$

D. 3

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$$(x+y)^4 = (x+y)(x+y)\,(x+y)\,(x+y)$$

$$= xxxx + yyyy + xyxy + yxyy + \ \ldots$$

Binomial Theorem: Idea

$$(x+y)^n = (x+y)(x+y)(x+y)\cdots(x+y)$$

Each term is of the form $x^k y^{n-k}$, since each term is is made by multiplying exactly n variables, either x or y.

How many times do we get $x^k y^{n-k}$? The number of ways to choose k of the n variables we multiple to be an x (the rest will be y).

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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Example – Word Permutations

How many ways to re-arrange the letters in the word "MATH"?

Poll:

A. $\binom{26}{4}$

B. 4^4

C. $4!$

D. I don't know

MATH

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Example – Word Permutations

How many ways to re-arrange the letters in the word "MUUMUU"?

____ ____ ____ ____ ____ ____

Example – Word Permutations

How many ways to re-arrange the letters in the word "MUUMUU"?

Choose where the 2 M's go, and then the U's are set **OR**
Choose where the 4 U's go, and then the M's are set

Either way, we get $\binom{6}{2} \cdot \binom{4}{4} = \binom{6}{4} \cdot \binom{2}{2} = \frac{6!}{2!4!}$

Another way to think about it

How many ways to re-arrange the letters in the word "MUUMUU"?

Arrange the 6 letters as if they were distinct.

$M_1U_1U_2M_2U_3U_4$

Then divide by 4! to account for duplicate M's and divide by 2! to account for duplicate U's.

Yields $\frac{6!}{2!4!}$

Another example – Word Permutations

How many ways to re-arrange the letters in the word "GODOGGY"?

Poll:

A. $7!$

B. $\frac{7!}{3!}$

C. $\frac{7!}{3!2!1!1!}$

D. $\binom{7}{3} \cdot \binom{5}{2} \cdot 3!$

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Combinatorial proof: Show that $M = N$

- Let S be a set of objects
- Show how to count $|S|$ one way => $|S| = M$
- Show how to count $|S|$ another way => $|S| = N$
- Conclude that $M = N$

Binomial Coefficient – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Pascal's Identity

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Follows from Binomial theorem

Pascal's Identities

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ years\ later\ \ldots$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

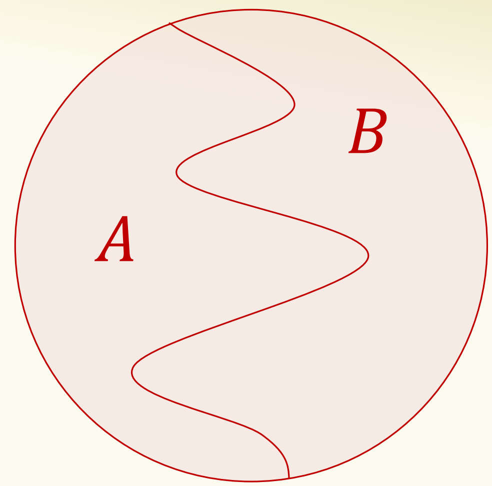
Hard work and not intuitive

Let's see a combinatorial argument

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| \quad = \quad |A| \quad + \quad |B|$$



$S = A \cup B$, disjoint

S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$ ➔ $|S| = \binom{n}{k}$

A: the set of size k subsets of $[n]$ including n

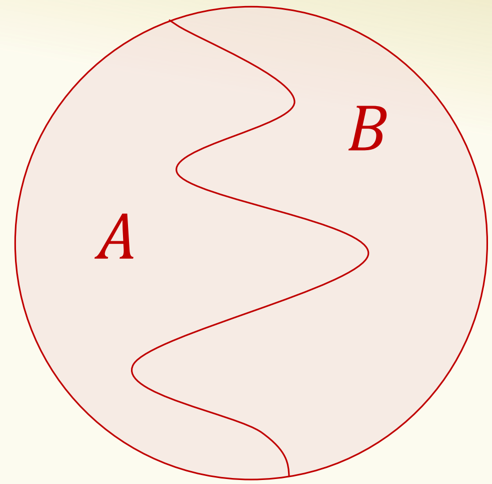
B: the set of size k subsets of $[n]$ NOT including n

Sum rule:
$|A \cup B| = |A| + |B|$

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| \quad = \quad |A| \quad + \quad |B|$$



S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$ ➔ $|S| = \binom{n}{k}$

e.g.: $n = 4, \quad S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

A: the set of size k subsets of $[n]$ including n

$$A = \{\{1,4\}, \{2,4\}, \{3,4\}\}. \qquad n = 4$$

B: the set of size k subsets of $[n]$ NOT including n

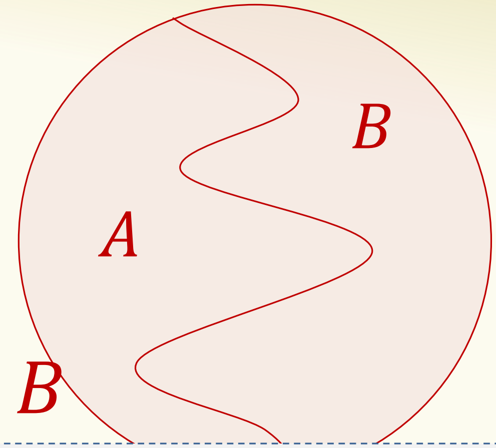
$$B = \{\{1,2\}, \{1,3\}, \{2,3\}\}$$

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S|$ $\quad$ $|A|$ $\quad$ $|B|$

$S = A \cup B$



S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$

A: the set of size k subsets of $[n]$ including n

B: the set of size k subsets of $[n]$ NOT including n

n is in set, need to choose $k-1$ elements from $[n-1]$

$$|A| = \binom{n-1}{k-1}$$

n not in set, need to choose k elements from $[n-1]$

$$|B| = \binom{n-1}{k}$$

combinatorial argument/proof

- Elegant
- Simple
- Intuitive

Algebraic argument

- Brute force
- Less Intuitive

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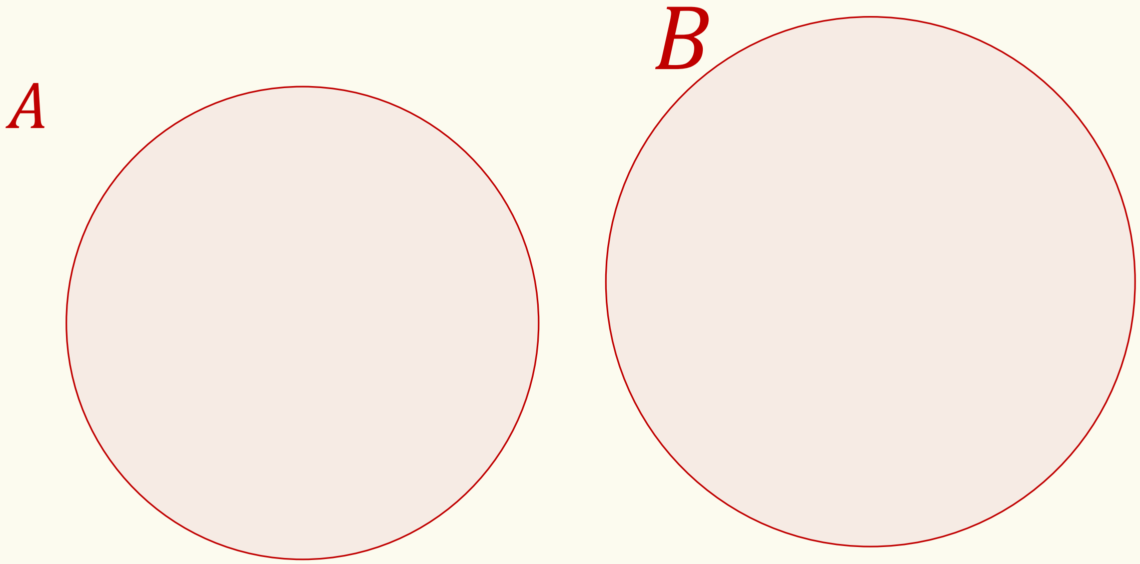
Recap Disjoint Sets

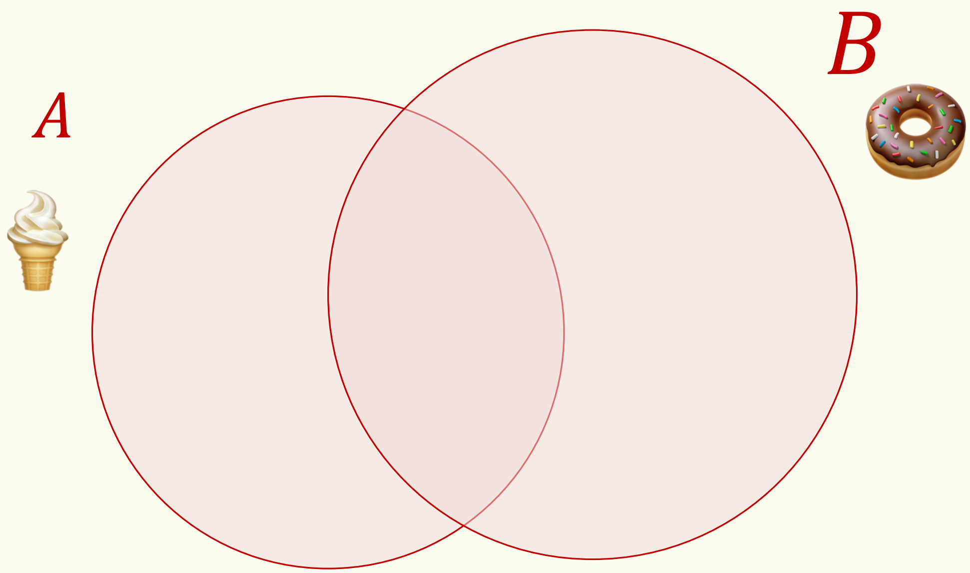
Sets that do not contain common elements ($A \cap B = \emptyset$)



Sum Rule: $|A \cup B| = |A| + |B|$

Inclusion-Exclusion

But what if the sets are not disjoint?



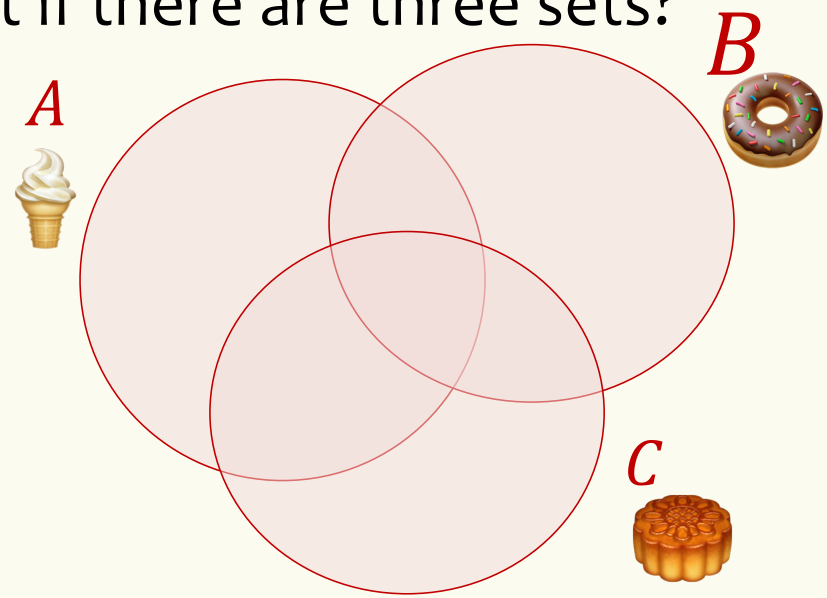
$|A| = 43$
$|B| = 20$
$|A \cap B| = 7$
$|A \cup B| = ???$

Fact. $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion

Not drawn to scale

What if there are three sets?



$|A| = 43$
$|B| = 20$
$|C| = 35$
$|A \cap B\ | = 7$
$|A \cap C\ | = 16$
$|B \cap C\ | = 11$
$|A \cap B \cap C\ | = 4$
$|A \cup B \cup C| = ???$

Fact.

$$
\begin{aligned}
|A \cup B \cup C| = {} & |A| + |B| + |C| \\
& - |A \cap B| - |A \cap C| - |B \cap C| \\
& + |A \cap B \cap C|
\end{aligned}
$$

Inclusion-Exclusion

Let A, B be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, if $A_1, A_2, \dots, A_n$ are sets, then

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| &= singles\ - doubles + triples\ - quads +\ \dots \\ &= (|A_1| + \cdots + |A_n|)\ - (|A_1 \cap A_2| +\ \dots + |A_{n-1} \cap A_n|) +\ \dots\end{aligned}$$

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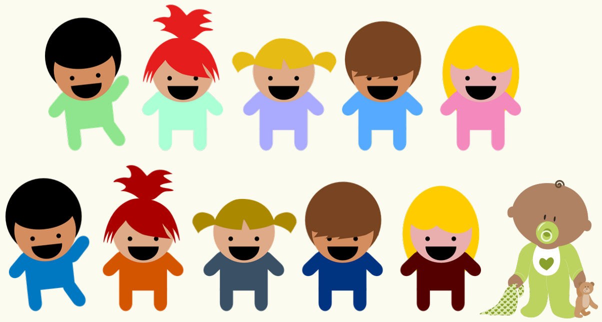
Pigeonhole Principle (PHP): Idea

10 pigeons, 9 pigeonholes

Pigeonhole Principle: Idea



If 11 children have to share 3 cakes, at least one cake must be shared by how many children?

Pigeonhole Principle – More generally

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\frac{n}{k}$ pigeons!

Proof. Assume there are $< \frac{n}{k}$ pigeons per hole.

Then, there are $< k\frac{n}{k} = n$ pigeons overall.

Contradiction!

Pigeonhole Principle – Better version

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

Reason. Can't have fractional number of pigeons

Syntax reminder:

- Ceiling: $\lceil x \rceil$ is x rounded up to the nearest integer (e.g., $\lceil 2.731 \rceil = 3$)
- Floor: $\lfloor x \rfloor$ is x rounded down to the nearest integer (e.g., $\lfloor 2.731 \rfloor = 2$)

Pigeonhole Principle – Example

In a room with 367 people, there are at least two with the same birthday.

Solution:

1. **367** pigeons = people
2. **365** holes = possible birthdays
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday

Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeonhole Principle – Example (Surprising?)

In every set S of 100 integers, there are at least ***two*** *elements whose difference is a multiple of 37.*

When solving a PHP problem:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

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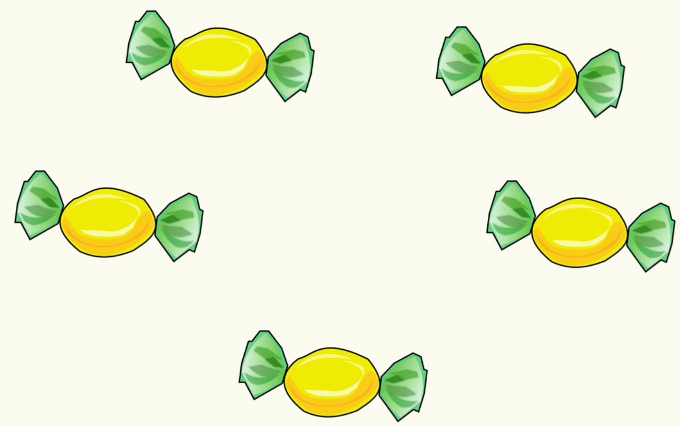
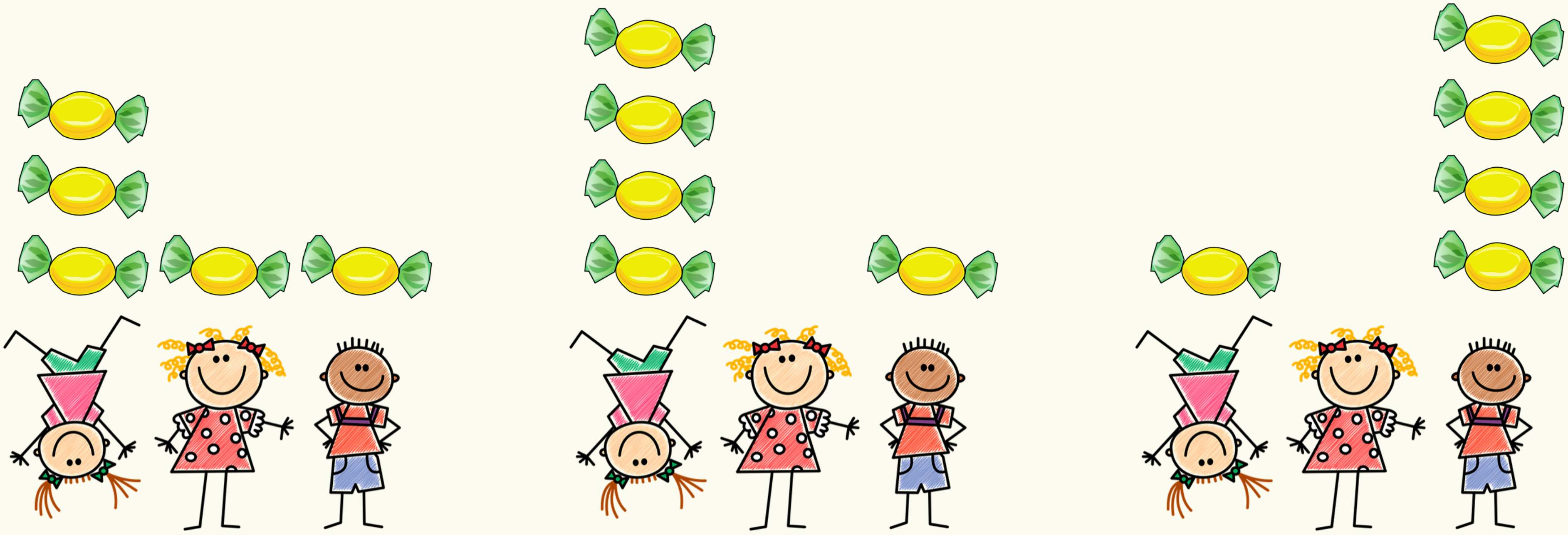
Example: Kids and Candies

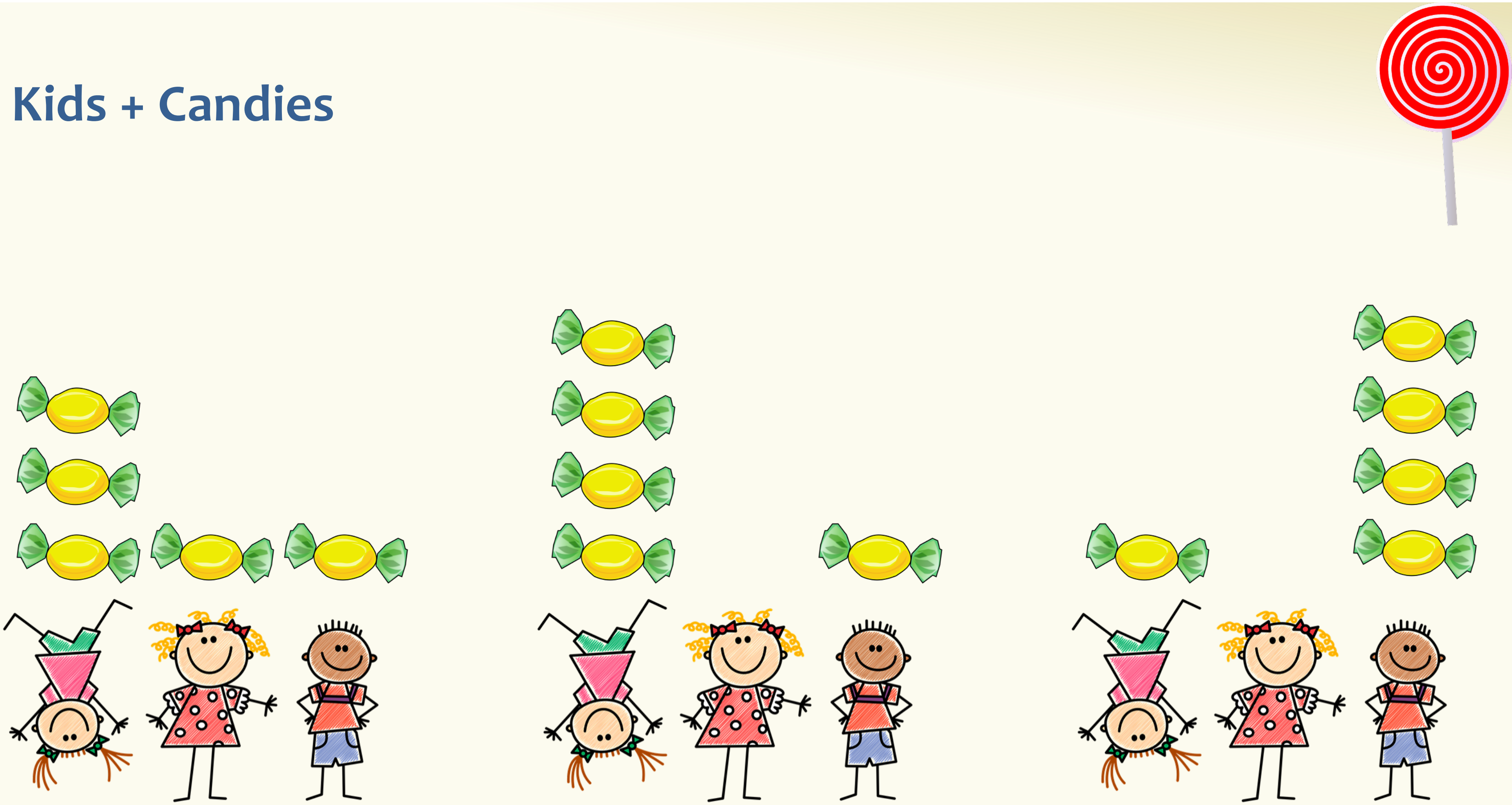


How many ways can we give five **indistinguishable** candies to these three kids?

Kids + Candies

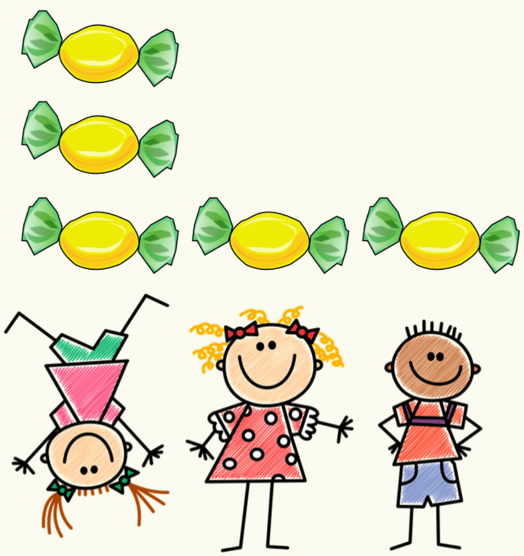


Kids + Candies



Kids + Candies

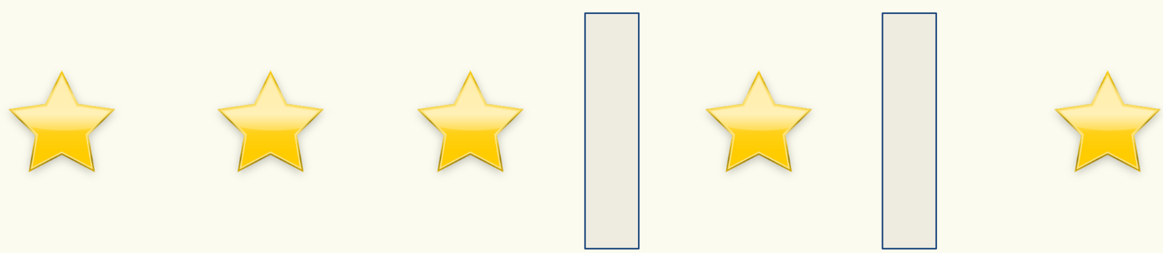
- Idea: count something different

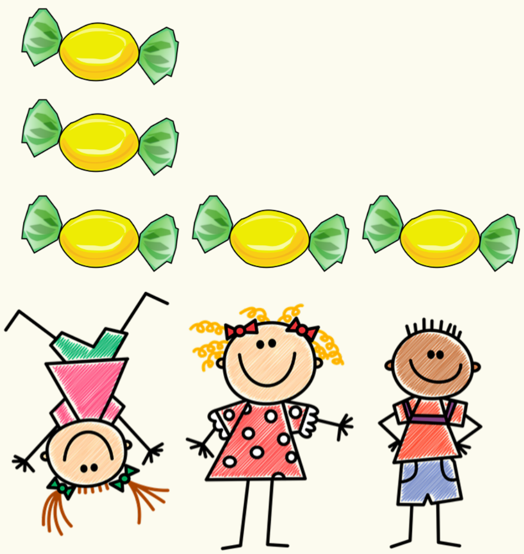


Kids + Candies

Idea: Count something equivalent

5 “stars” for candies, 2 “bars” for dividers.





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Idea: Count something equivalent

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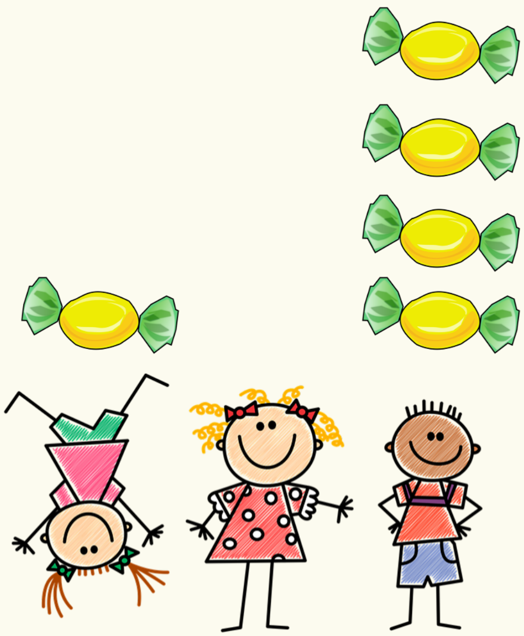
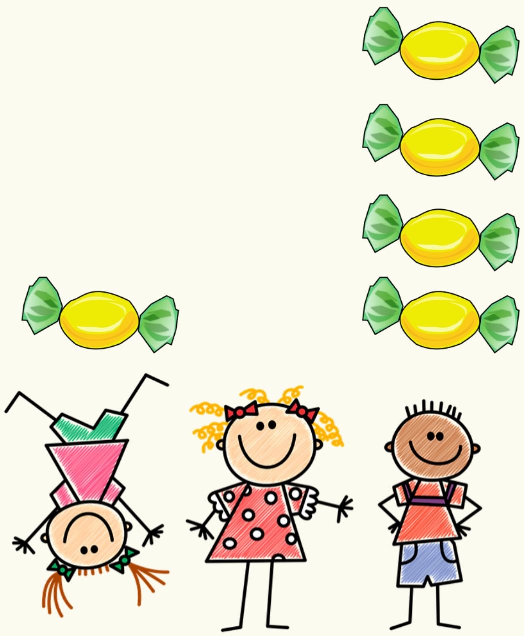


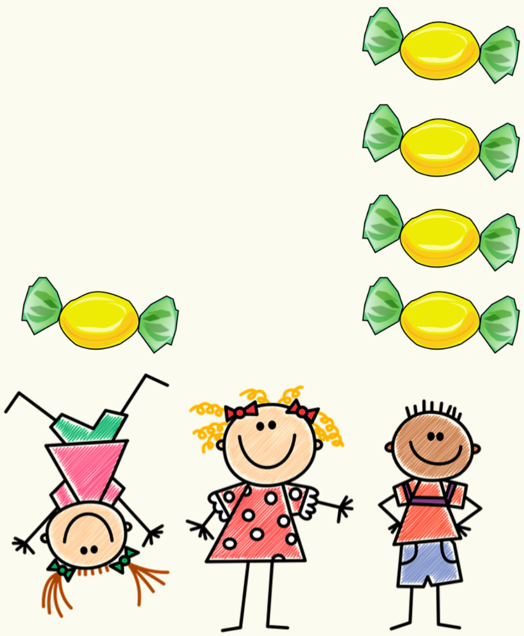
Kids + Candies



For each candy distribution, there is exactly one corresponding way to arrange the stars and bars.

Conversely, for each arrangement of stars and bars, there is exactly one candy distribution it represents.

Kids + Candies



Hence, the number of ways to distribute candies to the 3 kids is the number of arrangements of stars and bars.

This is

$$\binom{7}{2} = \binom{7}{5}$$

Stars and Bars / Divider method

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$