CSE 312
Foundations of Computing II

Lecture 20: Joint Distributions

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

$h: U \rightarrow \{0,1,\ldots,m-1\}$

Hash functions – few more comments

Approach: define a family $\mathcal{H}$ of hash fns s.t.

(a) if we pick hash fn $h \in \mathcal{H}$ u.a.r., it will behave well

$\forall x \in U \;\; \forall i \in \{0,\ldots,m-1\} \quad \Pr_{h \in \mathcal{H}}(h(x)=i) = \frac{1}{m}$

$\forall x,y \in U, x \neq y \;\; \forall i,j \in \{0,\ldots,m-1\} \quad \Pr_{h \in \mathcal{H}}(h(x)=i, h(y)=j) = \frac{1}{m^2}$

b) $\forall x \in U$, $\forall h \in \mathcal{H}$, $h(x)$ is efficiently computable $O(1)$ time

U → table of size m.

A class of hash fns that satisfies this

$|U| = N \qquad N \leq p$ prime #

$h_{a,b}(x) = (ax+b) \bmod p \bmod m$

$\mathcal{H} = \{ h_{a,b} \mid 1 \leq a \leq p-1, \; 0 \leq b \leq p-1 \}$

One of many constructions.

a_1, b_1

a_2, b_2

Agenda

- Joint Distributions
  - Cartesian Products
  - Joint PMFs/PDFs/CDFs and Joint Range
  - Marginal Distributions, etc.

Why joint distributions?

$(X_1, X_2, \ldots, X_n)$
3 4

- Given all of its user's ratings for different movies, and any preferences you have expressed, Netflix wants to recommend a new movie for you.

- Given a bunch of medical data correlating symptoms and personal history with diseases, predict what is ailing a person with a particular medical history and set of symptoms. $Pr(D \mid \text{Fever} > 100, BB = \ldots)$

- Given current traffic, pedestrian locations, weather, lights, etc. decide whether a self-driving car should slow down or come to a stop.

Review Cartesian Product

Definition. Let A and B be sets. The **Cartesian product** of A and B is denoted

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Example.

$$\underbrace{\{1,2,3\}}_{A} \times \underbrace{\{4,5\}}_{B} = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$$

$A \times B$

If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

The sets don't need to be finite! You can have $\mathbb{R} \times \mathbb{R}$ (often denoted $\mathbb{R}^2$)

Joint PMFs and Joint Range

Definition. Let X and Y be discrete random variables. The **Joint PMF** of X and Y is

and

$$p_{X,Y}(a,b) = \Pr(X = a, Y = b)$$

Definition. The **joint range** of $p_{X,Y}$ is

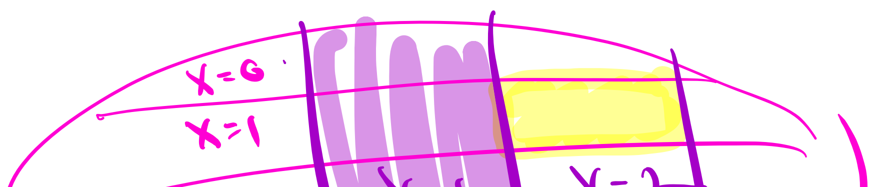
$$\boxed{\Omega(X,Y)} = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega(X)\times\Omega(Y)$$

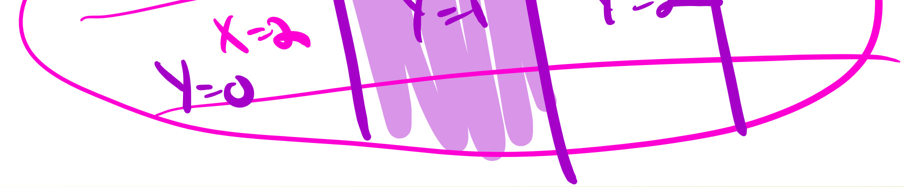
Ω_X

Note that

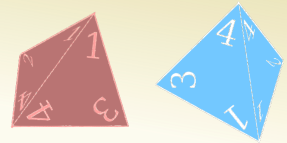
S

$$\sum_{(s,t)\in\Omega(X,Y)} p_{X,Y}(s,t) = 1$$





Example: Weird Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die.

$\Omega(X) = \{1,2,3,4\}$ and $\Omega(Y) = \{1,2,3,4\}$

In this problem, the joint PMF is

$$p_{X,Y}(x,y) = \begin{cases} 1/16, & x,y \in \Omega(X,Y) \\ 0, & \text{otherwise} \end{cases}$$

X\Y	1	2	3	4
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	$\boxed{1/16}$
4	1/16	1/16	1/16	1/16

and the joint range is (since all combinations have non-zero probability)

$\Omega(X,Y) = \Omega(X)\times\Omega(Y)$

Independence

Definition. Let X and Y be discrete random variables. The **Joint PMF** of X and Y is

$$p_{X,Y}(a,b) = \Pr(X = a, Y = b)$$

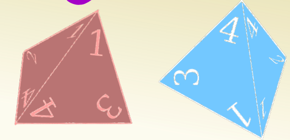
Definition. The **joint range** of $p_{X,Y}$ is

$$\Omega(X,Y) = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega(X) \times \Omega(Y)$$

Definition. X and Y are **independent** iff for all a, b

$$\Pr(X = a, Y = b) = \Pr(X = a) \cdot \Pr(Y = b)$$

Example: Weirder Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega(U) = \{1,2,3,4\}$ and $\Omega(W) = \{1,2,3,4\}$

~~$U=3$ $W=2$~~

$$\Omega(U, W) = \{(u, w) \in \Omega(U)\times\Omega(W) : u \leq w\} \neq \Omega(U)\times\Omega(W)$$

Poll:

What is $p_{U,W}(1, 3) = \Pr(U = 1, W = 3)$?

a. $1/16$

b. $2/16$

c. $1/2$

d. Not sure

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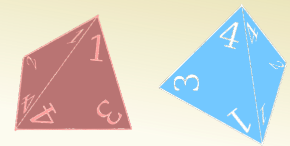
U\W	1	2	3	4
1				
2				
3				
4				

$\Pr(X=1, Y=3) + \Pr(X=3, Y=1)$

Example: Weirder Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega(U) = \{1,2,3,4\}$ and $\Omega(W) = \{1,2,3,4\}$

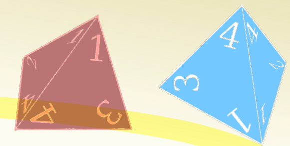
$$\Omega(U, W) = \{(u, w) \in \Omega(U)\times\Omega(W) : u \le w\} \neq \Omega(U)\times\Omega(W)$$

The joint PMF $p_{U,W}(u, w) = \Pr(U = u, W = w)$ is

$$p_{U,W}(u,w) = \begin{cases} 2/16, & (u,w) \in \Omega(U)\times\Omega(W) \text{ where } w > u \\ 1/16, & (u,w) \in \Omega(U)\times\Omega(W) \text{ where } w = u \\ 0, & \text{otherwise} \end{cases}$$

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Example: Weirder Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

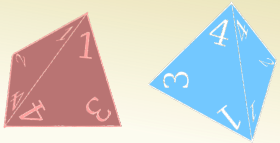
Suppose we didn't know how to compute $\Pr(U = u)$ directly. Can we figure it out if we know $p_{U,W}(u, w)$?

$$p_U(u) = \begin{cases} & u = 1 \\ & u = 2 \\ & u = 3 \\ & u = 4 \end{cases}$$

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

$\Pr(U = 1)$

Example: Weirder Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

Suppose we didn't know how to compute $\Pr(U = u)$ directly. Can we figure it out if we know $p_{U,W}(u, w)$?

$$p_U(u) = \begin{cases} 7/16, & u = 1 \\ 5/16, & u = 2 \\ 3/16, & u = 3 \\ 1/16, & u = 4 \end{cases}$$

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Marginal PMF

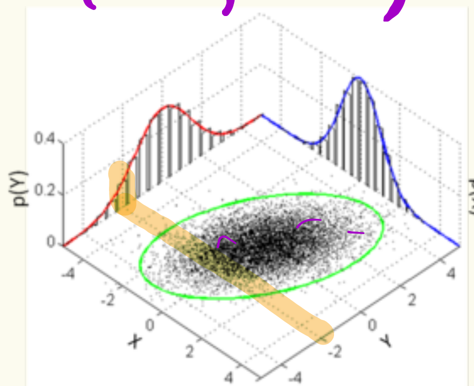
Definition. Let X and Y be discrete random variables and $p_{X,Y}(a,b)$ their joint PMF. The **marginal PMF** of X

$$p_X(a) = \boxed{\sum_{b \in \Omega(Y)}} p_{X,Y}(a,b)$$

$Pr(X=a)$ $Pr(X=a, Y=b)$

Similarly, $p_Y(b) = \sum_{a \in \Omega(X)} p_{X,Y}(a,b)$

Visual (for continuous X and Y)

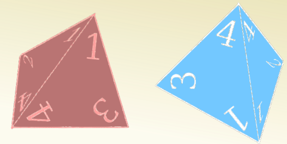


Joint Expectation

Definition. Let X and Y be discrete random variables and $p_{X,Y}(a,b)$ their joint PMF. The **expectation** of some function $g(x,y)$ with inputs X and Y

$$E[g(X,Y)] = \sum_{a \in \Omega(X)} \sum_{b \in \Omega(Y)} g(a,b) p_{X,Y}(a,b)$$

Another example.



Suppose the table below gives us the joint pmf of X and Y.

What is the marginal pmf of X? What is the marginal pmf of Y?
Are X and Y independent?
What is E(XY)?

$$Pr(X=x) = \begin{cases} 0.5 & x=1 \\ 0.5 & x=2 \end{cases}$$

$$Pr(Y=y) = \begin{cases} 0.5 & y=1 \\ 0.5 & y=2 \end{cases}$$

$$\underbrace{Pr(X=1, Y=1)}_{0.4} \neq \underbrace{P(X=1)}_{\frac{1}{2}} \cdot \underbrace{P(Y=1)}_{\frac{1}{2}}$$

X\Y	1	2
1	0.4	0.1
2	0.1	0.4

$i+j$ trials $\rightarrow p$. $Pr(i$ US requests$)$

- Suppose the number of requests Z to a particular web server per hour is Poisson(λ). And that the request comes from within the US with probability p.
- Let X be the number of requests per hour from the US and let Y be the number of requests per hour from outside the US. What is the joint pmf of X and Y? Are they independent?

$Z \sim Poi(\lambda)$

$\Omega_X = \{0,1,2,\ldots\}$
$\Omega_Y = \{0,1,2,\ldots\}$

$\Rightarrow X \sim Poi(\lambda p)$
$Y \sim Poi(\lambda(1-p))$

$Pr(X=i, Y=j)$

$\overset{L.T.P.}{=} \sum_{k=0}^{\infty} Pr(X=i, Y=j \mid Z=k) Pr(Z=k)$ $= Pr(X=i, Y=j \mid Z=i+j) Pr(Z=i+j)$

if $k \neq i+j$

$\binom{i+j}{i} p^i (1-p)^j$ $\quad e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!}$

$= \frac{\cancel{(i+j)!}}{i!\,j!\,\cancel{(i+j)!}} (\lambda p)^i (\lambda(1-p))^j e^{-\lambda}$

$= \frac{(\lambda p)^i}{i!} \frac{(\lambda(1-p))^j}{j!} e^{-\lambda(p+1-p)}$

$= e^{-\lambda p} \frac{(\lambda p)^i}{i!} \cdot e^{-\lambda(1-p)} \frac{[\lambda(1-p)]^j}{j!}$

$Pr(X=i) = \sum_{j=0}^{\infty} P(X=i, Y=j)$

$j=0$

$= P(X=i, Y=j)$

$\Pr(X=i)\Pr(Y=j)$

$= e^{-\lambda p}\frac{(\lambda p)^i}{i!}$

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X=x, Y=y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t<x, s<y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$

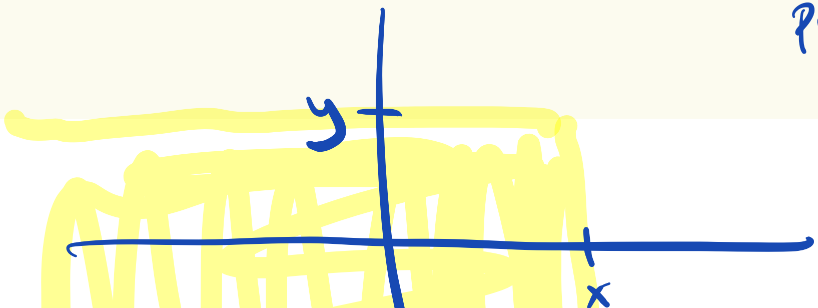
$f_X(x)\,dx \approx \Pr(X$ is within dx of $x)$

$f_{X,Y}(x,y)\,dx\,dy \approx \Pr(X$ is within dx of x, Y is within dy of $y)$

$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

$\Pr(X \le x, Y \le y) = F_{X,Y}(x,y)$



Independence (continuous random variables)

Definition. Let X and Y be continuous random variables. The **joint pdf** of X and Y is

$$f_{X,Y}(a,b) \neq \Pr(X = a, Y = b)$$

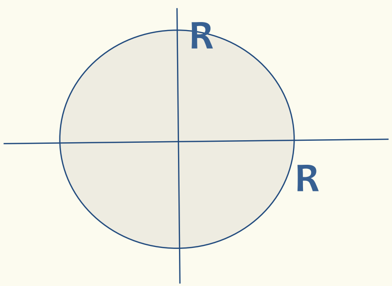
Definition. The **joint range** of $p_{X,Y}$ is

$$\Omega(X,Y) = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega(X) \times \Omega(Y)$$

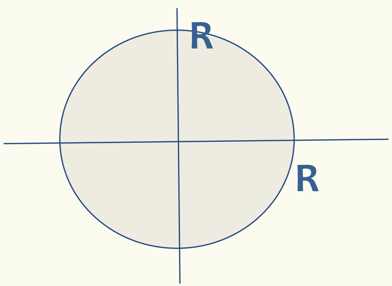
Definition. X and Y are **independent** iff for all a, b

$f_{X,Y}(a,b) = f_X(a) \cdot f_Y(b)$

- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - What is their joint density f(x,y)?



- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - What is the range of X & Y and the marginal density of X and of Y?



Poll:

What is Ω_X?

a. $[-\sqrt{R^2 - x^2}\,, \sqrt{R^2 - x^2}]$
b. $[-R\,, R]$
c. $[-\sqrt{R^2 - y^2}\,, \sqrt{R^2 - y^2}]$
d. Not sure

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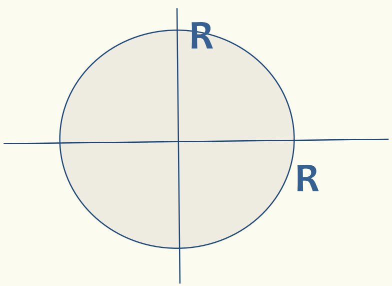
- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - Are X and Y independent?



Poll:
Are X and Y independent?
a. yes
b. no

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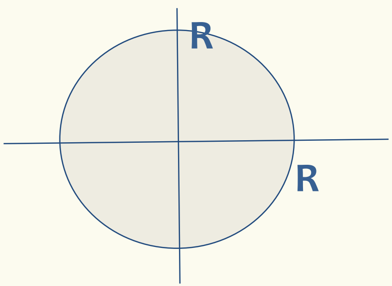
- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - What is E(Z)?



All of this generalizes to more than 2 random variables

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x, s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$

Brain Break