CSE 312

Foundations of Computing II

Lecture 21: Cont. Joint Distributions, Law of Total Expectation

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

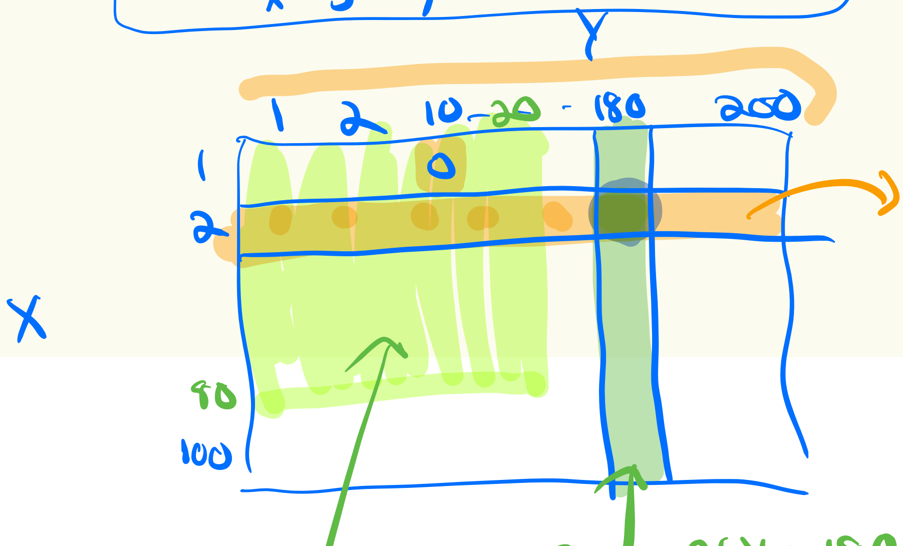
Agenda

- Continuous joint distributions
- Conditional Expectation and Law of Total Expectation

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X=x, Y=y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x, s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$

dxdy

$$\boxed{\sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} p_{XY}(x,y) = 1}$$



$\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$

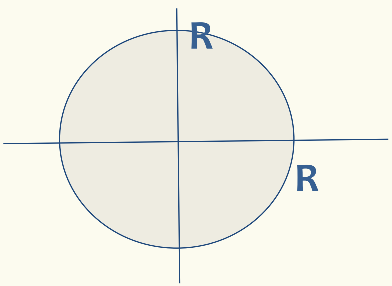
X, Y not indep.

$F_{XY}(80,20)$

$p(Y=180$

- Suppose that the surface of a disk is a circle with ~~area~~ radius R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - What is their joint density f(x,y)?

$$f_{XY}(x,y) = \begin{cases} c & x^2+y^2 \le R^2 \\ 0 & \text{o.w.} \end{cases}$$



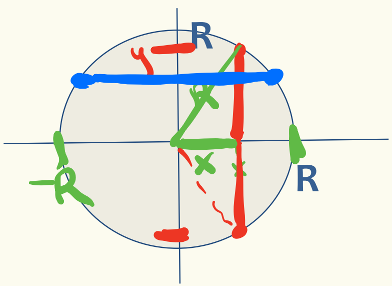
$$\iint_{x^2+y^2 \le R^2} c \, dx\,dy = 1$$

$$c\pi R^2 = 1$$

$$\Rightarrow c = \frac{1}{\pi R^2}$$

- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  – What is the range of X & Y and the marginal density of X and of Y?

$\Omega_X = [-R, R]$. $\quad \Omega_Y = [-R, R]$



$$f_X(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\, dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2}$$

$$f_X(x) = \begin{cases} \dfrac{2\sqrt{R^2-x^2}}{\pi R^2} & x^2 \le R^2 \\ 0 & \text{o.w.} \end{cases}$$

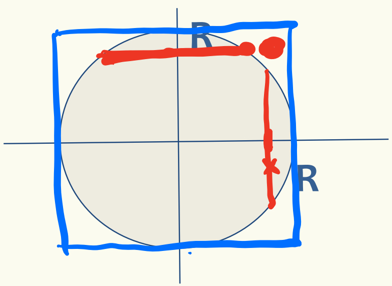
$$f_Y(y) = \begin{cases} \frac{2\sqrt{R^2-y^2}}{\pi R^2} & y^2 \le R^2 \\ 0 & o.w. \end{cases}$$

- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - Are X and Y independent?

$$\underline{\Omega_{X,Y} = \Omega_X \times \Omega_Y}$$

$$\Omega_X \times \Omega_Y$$

$$f_X(x) > 0$$

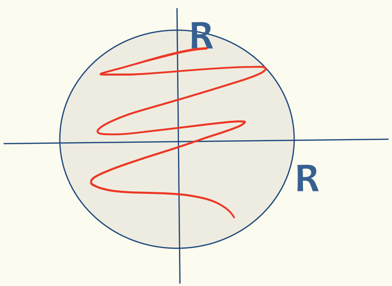


$$\underline{f_{XY}(x,y) = 0} \quad \text{by } \underline{f_X(x), f_Y(y) > 0}$$

- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - What is E(Z)?

$$Z = \sqrt{X^2+Y^2}$$

$$Z = g(X,Y)$$



$$\iint_{x^2+y^2\le R^2} \sqrt{x^2+y^2}\cdot\frac{1}{\pi R^2}\,dx\,dy$$

$$\iint_{A_{xy}} g(x,y)\, f(x,y)\,dx\,dy$$

All of this generalizes to more than 2 random variables

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Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{f_{XYZ}(x,y,z)}\, dx\, dy\, dz = 1$$

$$f_{X,Y}(x,y) = \int_{-\infty}^{\infty} f(x,y,z)\, dz.$$

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\, dy\, dz$$

Agenda

- Continuous joint distributions
- Conditional Expectation and Law of Total Expectation

Conditional Expectation

Definition. Let X be a discrete random variable then the **conditional expectation** of X given event A is

$$E[X \mid A] = \sum_{x \in \Omega(X)} x \Pr(X = x \mid A)$$

- Linearity of expectation still applies here

$$E[aX + bY + c \mid A] = aE[X \mid A] + bE[Y \mid A] + c$$

Conditional Expectation

$A = Y = y$

Definition. Let X be a discrete random variable then the **conditional expectation** of X given event $Y = y$ is

$$\underline{E[X \mid Y = y]} = \sum_{x \in \Omega(X)} x \underbrace{\Pr(X = x \mid Y = y)}_{\text{conditional prob mass fn. of } X \mid Y=y}$$

$$\underbrace{\sum_x P(X=x \mid Y=y)}_{\text{check}} = 1$$

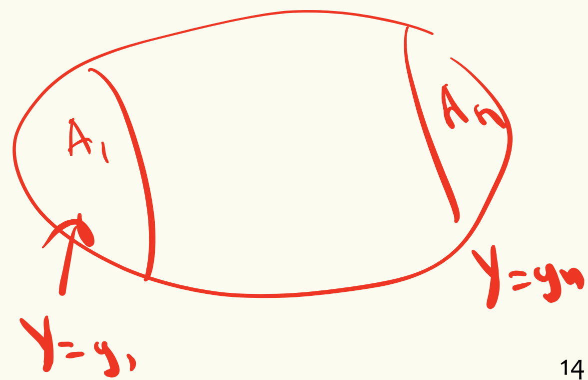
- Linearity of expectation still applies here

$$E[aX + bY + c \mid Y = y] = aE[X \mid Y = y] + bE[Y \mid Y = y] + c$$

Law of Total Expectation

Law of Total Expectation (event version). Let X be a random variable and let events $A_1, \ldots, A_n$ partition the sample space. Then,

$$E[X] = \sum_{i=1}^{n} E[X|A_i]\Pr(A_i)$$



Proof of Law of Total Expectation

Follows from Law of Total Probability and manipulating sums

$$E[X] = \sum_{x \in \Omega(X)} x \Pr(X = x)$$

$$= \sum_{x \in \Omega(X)} x \sum_{i=1}^{n} \Pr(X = x \mid A_i)\Pr(A_i) \qquad \text{(by LTP)}$$

$$= \sum_{i=1}^{n} \Pr(A_i) \left[\sum_{x \in \Omega(X)} x \Pr(X = x \mid A_i)\right] \qquad \text{(change order of sums)}$$

$$= \sum_{i=1}^{n} \Pr(A_i)\, E[X \mid A_i] \qquad \text{(def of cond. expect.)}$$

Law of Total Expectation

Law of iterated exps.

Law of Total Expectation (random variable version). Let X be a random variable and Y be a discrete random variable. Then,

$$E[X] = \sum_{y \in \Omega(Y)} E[X|Y = y]\Pr(Y = y)$$

$$E[\underbrace{E(X|Y)}_{\text{random variable}}]$$

random variable takes

$y \in \Omega_Y$

$E(X|Y=y)$ w.p. $\Pr(Y=y)$

$\Omega_Y = \{1,2,3\}$

$$E(X|Y) = \begin{cases} E(X|Y=1) & \Pr(Y=1) \\ E(X|Y=2) & P(Y=2) \\ E(X|Y=3) & P(Y=3 \end{cases}$$

Example: Flipping Coins

Suppose wanted to analyze flipping a random number of coins. Suppose someone gave us $Y \sim Poi(5)$ fair coins and we wanted to compute the expected number of heads X from flipping those coins.

Example: Computer Failures

Suppose your computer operates in a sequence of steps, and that at each step i your computer will fail with probability p (independently of other steps). Let X be the number of steps it takes your computer to fail. What is $E[X]$?

$$Y = \begin{cases} 1 & \text{if first trial failure} \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \underbrace{E(X \mid Y=1)}_{1} \underbrace{P(Y=1)}_{\cdot\, p} + E(X \mid Y=0)\, P(Y=0)$$

1 + # additional trials/steps to failure.

$$E(1 + X \mid Y=0)\,(1-p)$$

$$1 + E(X)$$

$$E(X) = p + (1-p)\big(1 + E(X)\big)$$

$$= \underbrace{p + 1-p}_{1} + \big((1-p)E(X)\big)$$

$$pE(X) = 1$$

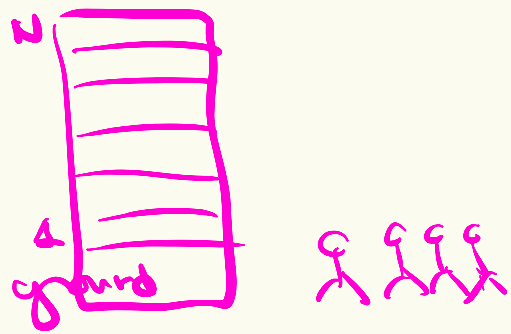
$$E(X) = \frac{1}{p}$$

Elevator rides

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where others get off, compute the expected number of stops the elevator will make before discharging all the passengers.

Y : # stops elevator makes.

$$E(Y) = \sum_{k=0}^{\infty} E(Y \mid X=k)\, \underbrace{P(X=k)}_{e^{-10}\frac{10^k}{k!}}$$



X : # people entering

X ~ Poisson(10)

$$Y = Y_1 + Y_2 + \dots + Y_n$$

$$Y_i = \begin{cases} 1 & \text{stops on } i^{th} \text{ floor} \\ 0 & \text{o.w.} \end{cases}$$

$$E(Y \mid X=k) = E(Y_1 + Y_2 + \dots + Y_N \mid X=k)$$

$$\underset{\text{LOE for cond exps}}{=} \sum_{i=1}^{N} \underbrace{E(Y_i \mid X=k)}_{P(\text{step on } i^{th} \text{ floor} \mid X=k)}$$

Z_i: # of people who step on i^{th} floor $\mid$ $X=k$

$$\mathrm{Bin}\left(k, \frac{1}{N}\right)$$

$$\Pr(Z_i > 0) = 1 - \underbrace{\Pr(Z_i = 0)}_{\left(\frac{N-1}{N}\right)^k}$$

Reference Sheet (with continuous RVs)

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Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x} \sum_{s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\,ds\,dt$
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Expectation	$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$
Conditional PMF/PDF	$p_{X \mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$E[X \mid Y = y] = \sum_{x} x p_{X \mid Y}(x \mid y)$	$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X \mid Y}(x \mid y)\,dx$
Independence	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$