CSE 312

Foundations of Computing II

Lecture 21: Cont. Joint Distributions, Law of Total Expectation

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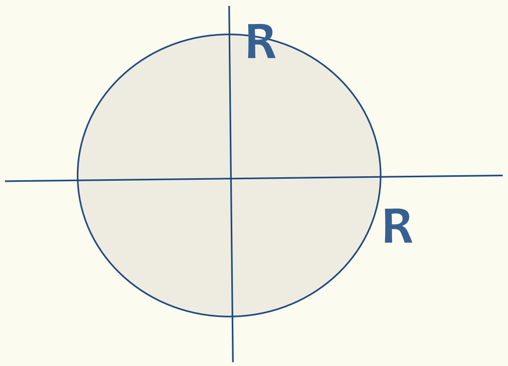
Slide Credit: Based on Stefano Tessaro's slides for 312 19au incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Agenda

- Continuous joint distributions
- Conditional Expectation and Law of Total Expectation

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, dsdt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dxdy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dxdy$

- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  – What is their joint density f(x,y)?



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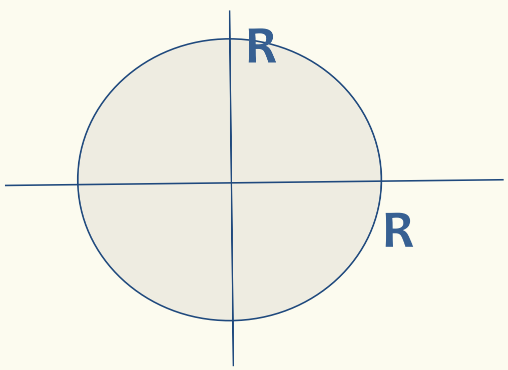
  - What is the range of X & Y and the marginal density of X and of Y?



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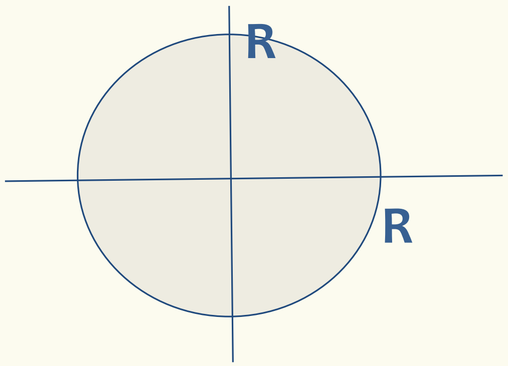
  - Are X and Y independent?



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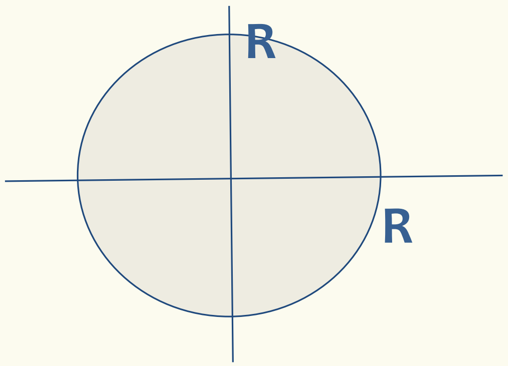
  - What is E(Z)?



All of this generalizes to more than 2 random variables

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Conditional Expectation

Definition. Let X be a discrete random variable then the **conditional expectation** of X given event A is

$$E[X \mid A] = \sum_{x \in \Omega(X)} x \Pr(X = x \mid A)$$

- Linearity of expectation still applies here

$$E[aX + bY + c \mid A] = aE[X \mid A] + bE[Y \mid A] + c$$

Conditional Expectation

Definition. Let X be a discrete random variable then the **conditional expectation** of X given event $Y = y$ is

$$E[X \mid Y = y] = \sum_{x \in \Omega(X)} x \Pr(X = x \mid Y = y)$$

- Linearity of expectation still applies here

$$E[aX + bY + c \mid Y = y] = aE[X \mid Y = y] + bE[Y \mid Y = y] + c$$

Law of Total Expectation

Law of Total Expectation (event version). Let X be a random variable and let events $A_1, \ldots, A_n$ partition the sample space. Then,

$$E[X] = \sum_{i=1}^{n} E[X|A_i]\Pr(A_i)$$

Proof of Law of Total Expectation

Follows from Law of Total Probability and manipulating sums

$$E[X] = \sum_{x \in \Omega(X)} x \Pr(X = x)$$

$$= \sum_{x \in \Omega(X)} x \sum_{i=1}^{n} \Pr(X = x \mid A_i) \Pr(A_i)$$ (by LTP)

$$= \sum_{i=1}^{n} \Pr(A_i) \sum\nolimits_{x \in \Omega(X)} x \Pr(X = x \mid A_i)]$$ (change order of sums)

$$= \sum_{i=1}^{n} \Pr(A_i)\, E[X \mid A_i]$$ (def of cond. expect.)

Law of Total Expectation

Law of Total Expectation (random variable version). Let X be a random variable and Y be a discrete random variable. Then,

$$E[X] = \sum_{y \in \Omega(Y)} E[X|Y = y]\Pr(Y = y)$$

Example: Flipping Coins

Suppose wanted to analyze flipping a random number of coins. Suppose someone gave us $Y \sim Poi(5)$ fair coins and we wanted to compute the expected number of heads X from flipping those coins.

Example: Computer Failures

Suppose your computer operates in a sequence of steps, and that at each step i your computer will fail with probability p (independently of other steps). Let X be the number of steps it takes your computer to fail. What is $E[X]$?

Elevator rides

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where others get off, compute the expected number of stops the elevator will make before discharging all the passengers.

Reference Sheet (with continuous RVs)

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Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x} \sum_{s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s) ds dt$
Normalization	$\sum_{x} \sum_{y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$
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Expectation	$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Conditional PMF/PDF	$p_{X \mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$E[X \mid Y = y] = \sum_{x} x p_{X \mid Y}(x \mid y)$	$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X \mid Y}(x \mid y) dx$
Independence	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$